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POINT-COUNTABLE COVERS AND SEQUENCE-COVERING MAPS

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Abstract. We answer some questions posed in the book [4] on the theory of generalized metric spaces.

1. Introduction

In the book [4] on the theory of generalized metric spaces, several questions concerning point-countable covers and sequence-covering maps are posed. In this paper, we answer some of them. The readers can refer to [4] (or, [5]) for the motivation and related matters of each question we answer.

All spaces are assumed to be regular T_1, unless a specific separation axiom is indicated. The symbol $\mathbb{N}$ (resp., $\mathbb{D}$) is the set of positive integers (resp., the set of 0 and 1). Let $\mathbb{S} = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be the usual convergent sequence. Let $S_\omega = \{\infty\} \cup \{(n, m) : n, m \in \mathbb{N}\}$ be the sequential fan, where each (n, m) is isolated in S_ω and a basic open neighborhood of ∞ is of the form $N(f) = \{\infty\} \cup \{(n, m) : n \in \mathbb{N}, m \geq f(n)\}$ for a function $f \in \mathbb{N}^{\mathbb{N}}$. In other words, S_ω is the quotient space obtained by identifying the limits of countably many convergent sequences. A family $\mathcal{P}$ of subsets of a space X is said to be *point-countable* (resp., *compact-finite*) if for each point $x \in X$ (resp., compact set $K \subset X$), the set $\{P \in \mathcal{P} : x \in P\}$ (resp., $\{P \in \mathcal{P} : P \cap K \neq \emptyset\}$) is countable (resp., finite). For a point $x \in X$ and a subset $P \subset X$, P is said to be a *sequential neighborhood* of x

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in X if $x \in P$ and every sequence in X converging to x is eventually in P (i.e., if $x_n \to x$, then $x_n \in P$ for all but finitely many $n \in \mathbb{N}$).

We recall some definitions [4].

Definition 1.1. Let $\mathcal{P}$ be a family of subsets in a space X.

(1) $\mathcal{P}$ is a *cfp-network* for X if whenever K is compact and V is open with $K \subset V \subset X$, there are finitely many $P_1, \dots, P_n \in \mathcal{P}$ and a closed cover $\{K_1, \dots, K_n\}$ of K such that $K \subset P_1 \cup \dots \cup P_n \subset V$ and $K_j \subset P_j$ for all j.

(2) $\mathcal{P}$ is a *k-network* for X if whenever K is compact and V is open with $K \subset V \subset X$, there are finitely many $P_1, \dots, P_n \in \mathcal{P}$ such that $K \subset P_1 \cup \dots \cup P_n \subset V$.

(3) $\mathcal{P}$ is a *cs-network* (resp., *cs*-network*) for X if whenever $x_n \to x$ and V is a neighborhood of x, there is some $P \in \mathcal{P}$ such that $x \in P$ and $x_n \in P$ for all but finitely many $n \in \mathbb{N}$ (resp., $x_n \in P$ for infinitely many $n \in \mathbb{N}$).

A k-network consisting of closed subsets is a cfp-network, and every cfp-network is a both k-network and cs^*-network. Every cs-network is a cs^*-network.

Definition 1.2. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets in a space X satisfying (a) for each $x \in X$, $x \in \bigcap \mathcal{P}_x$ and if V is a neighborhood of x, then there is some $P \in \mathcal{P}_x$ such that $x \in P \subset V$; (b) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak-base* for X if for every $G \subset X$, G is open in X whenever for each $x \in G$, there is some $P \in \mathcal{P}_x$ with $P \subset G$.

(2) $\mathcal{P}$ is an *sn-network* for X if for each $x \in X$, every member of $\mathcal{P}_x$ is a sequential neighborhood of x.

(3) A space X is *gf-countable* (resp., *snf-countable*) if it has a weak-base (resp., an sn-network) $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ such that each $\mathcal{P}_x$ is countable.

If $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a weak-base for a space X, using the condition (1) in Definition 1.2, we can see that each member of $\mathcal{P}_x$ is a sequential neighborhood of x. Hence, a weak-base is an sn-network. In particular, a gf-countable space is snf-countable.

Definition 1.3. Let $f : X \to Y$ be a continuous onto map.

(1) f is *1-sequence-covering* if for each $y \in Y$, we can take a point $x_y \in f^{-1}(y)$ satisfying that for each sequence $\{y_n : n \in \mathbb{N}\}$ in Y converging to a point in Y, there is a sequence $\{x_n : n \in \mathbb{N}\}$ in X converging to the point x_y such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$.

(2) f is *sequence-covering* if for each sequence $\{y_n : n \in \mathbb{N}\}$ in Y converging to a point in Y, there is a sequence $\{x_n : n \in \mathbb{N}\}$ in X converging to a point in X such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$.

(3) f is *sequentially quotient* if for each sequence $\{y_n : n \in \mathbb{N}\}$ in Y converging to a point in Y, there are a subsequence $\{y_{n_j} : j \in \mathbb{N}\}$ of $\{y_n\}$ and a sequence $\{x_j : j \in \mathbb{N}\}$ in X converging to a point in X such that $f(x_j) = y_{n_j}$ for all $j \in \mathbb{N}$.

(4) f is *pseudo-sequence-covering* if whenever S is a convergent sequence with its limit in Y, there is a compact set $K \subset X$ with $f(K) = S$.

(5) f is *boundary-compact* (resp., *at most boundary-one*) if the boundary of $f^{-1}(y)$ is compact (resp., at most one point) for each $y \in Y$.

(6) f is an *s-map* if $f^{-1}(y)$ is separable for each $y \in Y$.

Every 1-sequence-covering map is sequence-covering, and every sequence-covering map is both sequentially quotient and pseudo-sequence-covering.

2. Questions and Answers

It is known that a closed map of a regular T_1-space in which each point is a G_δ-set is sequentially quotient [4, Lemma 2.3.3]. Therefore, it is natural to consider the following question.

Question 2.1 ([4, Question 2.3.15])**.** Is a closed map of a T_2-space in which each point is a G_δ-set sequentially quotient?

This question is in the negative.

Proposition 2.2. *There is a closed map $\varphi : X \to \mathbb{S}$ which is not sequentially quotient such that X is T_2 (non-regular) and every point of X is a G_δ-set.*

Proof. Consider the Stone-Čech compactification $\beta\mathbb{N}$. For each $p \in \beta\mathbb{N} \setminus \mathbb{N}$, let $\mathcal{N}(p) = \{\{p\} \cup A : A \in p\}$, and τ be the topology generated by

$$\{\{n\} : n \in \mathbb{N}\} \cup \bigcup \{\mathcal{N}(p) : p \in \beta\mathbb{N} \setminus \mathbb{N}\}.$$

Then $X = (\beta\mathbb{N}, \tau)$ is a T_2-space such that each point is a G_δ-set. Let $\varphi : X \to \mathbb{S}$ be the map defined as follows: $\varphi(n) = 1/n$ if $n \in \mathbb{N}$, and $\varphi(p) = 0$ if $p \in \beta\mathbb{N} \setminus \mathbb{N}$. This map φ is obviously continuous and onto. Moreover, it is closed. Indeed, let A be a closed subset in X. If $A \setminus \mathbb{N} \neq \emptyset$, $\varphi(A)$ is obviously closed in $\mathbb{S}$. If $A \subset \mathbb{N}$, A must be a finite set, so $\varphi(A)$ is closed in $\mathbb{S}$. However, since every convergent sequence in X is a finite set, φ is not sequentially quotient. $\square$

Recall that every cfp-network is a cs^*-network.

Question 2.3 ([4, Question 2.5.21 (2)])**.** In ZFC, is there a regular T_1-space X which has a point-countable cs^*-network but no any point-countable cfp-network?

This question is in the negative.

Proposition 2.4. *There is a compact T_2-space X with a point-countable cs-network such that X does not have any point-countable cfp-network.*

PROOF. Let X be the Stone-Čech compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$. Every convergent sequence in X is a finite set, so $\mathcal{P} = \{\{x\} : x \in X\}$ is a point-countable cs-network (hence, cs^*-network) for X. Since a compact space with a point-countable k-network is metrizable [2], X does not have any point-countable k-network. Since a cfp-network is a k-network, X does not have any point-countable cfp-network. □

A continuous onto map $f : X \to Y$ is said to be *almost-open* if for each $y \in Y$, we can take a point $x_y \in f^{-1}(y)$ such that if U is a neighborhood of x_y in X, then $f(U)$ is a neighborhood of y in Y. An open map is almost-open. It is easy to see that every almost-open map of a first-countable space is sequence-covering. However, there is an open map of a Fréchet space onto $\mathbb{S}$ which is not sequence-covering: see Yanagimoto [11, Example 4.4]. A space X is said to be *strongly Fréchet* if for each point $x \in X$ and a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of X, $x \in \bigcap\{\overline{A_n} : n \in \mathbb{N}\}$ implies that there are points $x_n \in A_n$ such that $x_n \to x$ $(n \to \infty)$. Yanagimoto's Fréchet space is not strongly Fréchet and its cardinality is the continuum.

Question 2.5 ([4, Question 2.6.19])**.** Is each almost-open map of a strongly Fréchet space sequence-covering?

This question is in the negative. Using Nyikos' construction in [8], we give a counterexample of an open map. Let $2^{<\omega}$ be the full binary tree of height ω (i.e., the set of all finite sequences of 0's and 1's with the extension order $\subset$). We give a topology for the set $2^{<\omega} \cup \mathbb{D}^\omega$ as follows: every point of $2^{<\omega}$ is isolated, and a basic open neighborhood at $f \in \mathbb{D}^\omega$ is of the form $\{f\} \cup \{f \restriction n : n \geq k\}$, where $k \in \omega$ and $f \restriction n$ is the restriction of f to the domain n. Since $2^{<\omega} \cup \mathbb{D}^\omega$ is locally compact, there is the one-point compactification $2^{<\omega} \cup \mathbb{D}^\omega \cup \{\infty\}$. Let $S(\mathbb{D}^\omega) = 2^{<\omega} \cup \{\infty\}$ be the subspace of $2^{<\omega} \cup \mathbb{D}^\omega \cup \{\infty\}$. A basic open neighborhood at $\infty \in S(\mathbb{D}^\omega)$ is of the form $S(\mathbb{D}^\omega) \setminus (B_0 \cup \cdots \cup B_n)$, where each B_i is a branch (= a maximal chain) in $2^{<\omega}$. A space X is said to be *bisequential* if every ultrafilter $\mathcal{A}$ converging to a point $x \in X$ contains a decreasing sequence $\{A_n : n \in \omega\}$ converging to x [7]. Every first-countable space is bisequential, and every bisequential space is strongly Fréchet [7]. It is know that $S(\mathbb{D}^\omega)$ is bisequential: see the proof in [8, Corollary 2.8].

Theorem 2.6. *There is an open map $\varphi : X \to \mathbb{S}$ which is not sequence-covering such that X is countable and bi-sequential.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a cover of $2^{<\omega}$ consisting of branches in $2^{<\omega}$, where $B_n \neq B_m$ if $n \neq m$. We define a map $\varphi : S(\mathbb{D}^\omega) \to \mathbb{S}$ as follows: $\varphi(\infty) = 0$, and $\varphi(B_n \setminus (B_1 \cup \cdots \cup B_{n-1})) = \{1/n\}$ for each $n \in \mathbb{N}$. Obviously φ is continuous and onto. We see that φ is open. Let U be an open subset in $S(\mathbb{D}^\omega)$. If $\infty \notin U$, $\varphi(U)$ is obviously open in $\mathbb{S}$. If $\infty \in U$, without loss of generality, we may put $U = S(\mathbb{D}^\omega) \setminus (C_1 \cup \cdots \cup C_k)$, where each C_i is a branch in $2^{<\omega}$. We can take some $l \in \mathbb{N}$ such that for each $n \geq l$, $B_n \setminus (C_1 \cup \cdots \cup C_k)$ is infinite. This implies $\varphi(U) \supset \{0\} \cup \{1/n : n \geq l\}$. Thus φ is open.

Claim: If $b_n \in B_n$ $(n \in \mathbb{N})$ and $b_n \neq b_m$ for $n < m$, then $\{b_n : n \in \mathbb{N}\}$ contains an infinite chain.

Proof. Let $\mathrm{ht}(b_1, 2^{<\omega}) = k$ (i.e., b_1 has just k-many predecessors in $2^{<\omega}$). Fix an $l \in \mathbb{N}$ with $k < 2^l - 1$. Then $|\{s \in \mathrm{Lev}_{k+l}(2^{<\omega}) : b_1 \subset s\}| = 2^l$. For each $s \in \mathrm{Lev}_{k+l}(2^{<\omega})$ such that $b_1 \subset s$ and $s \notin B_1$, using the fact $2^{<\omega} = \bigcup\{B_n : n \in \mathbb{N}\}$, we can take $B_{n(s)} \in \{B_n : n \in \mathbb{N}\}$ with $s \in B_{n(s)}$. Then $\{b_1, b_{n(s)}\} \subset B_{n(s)}$ and one of these $b_{n(s)}$'s is not a predecessor of b_1. Therefore we can take some b_{n_1} with $b_1 \subset b_{n_1}$ $(n_1 \neq 1)$. Continuing this operation, we can obtain an infinite chain in $\{b_n : n \in \mathbb{N}\}$. □

Take any point $b_n \in \varphi^{-1}(1/n) \subset B_n$ for each $n \in \mathbb{N}$. Then, by Claim above, there is an infinite chain $\{b_{n_j} : j \in \mathbb{N}\} \subset \{b_n : n \in \mathbb{N}\}$. Since an infinite chain is contained in some branch, $\{b_{n_j} : j \in \mathbb{N}\}$ is closed in $S(\mathbb{D}^\omega)$. Hence, φ is not sequence-covering. □

If a map $f : X \to Y$ is 1-sequence-covering countable-to-one and X has a point-countable sn-network, then Y also has a point-countable sn-network [3].

Question 2.7 ([4, Question 2.6.21])**.** Let $f : X \to Y$ be a 1-sequence-covering, at most boundary-one, s-map and assume that X has a point-countable sn-network. Does Y have a point-countable sn-network?

This question is in the negative. Recall that a weak-base is an sn-network.

Theorem 2.8. *There is a 1-sequence-covering, at most boundary-one, s-map $\varphi : X \to Y$ such that X has a σ-point-finite weak-base, but Y does not have any point-countable sn-network.*

Proof. Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ such that $|\mathcal{A}| = \mathfrak{c}$. We put $\mathcal{A} = \{A_r : r \in \mathbb{C}\}$, where $\mathbb{C}$ is the Cantor set. Let $\Psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$ be the space with the topology: every point in $\mathbb{N}$ is isolated in $\Psi(\mathcal{A})$, and a basic open neighborhood of $A_r \in \Psi(\mathcal{A})$ is of the form $\{A_r\} \cup (A_r \setminus \{1, \cdots, n\})$ for $n \in \mathbb{N}$. This $\Psi(\mathcal{A})$ does not have any point-countable sn-network. Indeed, if there is a point-countable sn-network $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in \Psi(\mathcal{A})\}$, then $\{\operatorname{Int} P : P \in \mathcal{P}\}$ is a point-countable base of $\Psi(\mathcal{A})$, this is a contradiction. Let $X = \mathbb{C} \times \mathbb{S}$, and we give X a topology as follows: a basic open neighborhood of $(r, 1/n) \in \mathbb{C} \times (\mathbb{S} \setminus \{0\})$ has the form $U \times \{1/n\}$, where U is an open-and-closed neighborhood of $r \in \mathbb{C}$, and a basic open neighborhood of $(r, 0) \in \mathbb{C} \times \{0\}$ has the form

$$\{(r,0)\} \cup \left(\bigcup\{U_n \times \{1/n\} : n \in A_r, n \geq k\}\right),$$

where $k \in \mathbb{N}$ and U_n is an open-and-closed neighborhood of $r \in \mathbb{C}$. Obviously X is Tychonoff.

We see that X has a σ-point-finite weak-base (hence, a point-countable sn-network). For each $r \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$N(r,n) = \{(r,0)\} \cup \{(r, 1/k) : k \in A_r, k \geq n\} \text{ and } \mathcal{P}_{(r,0)} = \{N(r,n) : n \in \mathbb{N}\}.$$

Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base for $\mathbb{C} \times (\mathbb{S} \setminus \{0\})$, and for each $(r, 1/n) \in X$, let $\mathcal{P}_{(r,1/n)} = \{B \in \mathcal{B} : (r, 1/n) \in B\}$. Then obviously $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a weak-base for X. We see that $\mathcal{P}$ is σ-point-finite. Let $\mathcal{Q}_n = \{B_n\}$, and let $\mathcal{R}_n = \{N(r,n) : r \in \mathbb{C}\}$. Then trivially both $\mathcal{Q}_n$ and $\mathcal{R}_n$ are point-finite, and $\mathcal{P} = \bigcup\{\mathcal{Q}_n \cup \mathcal{R}_n : n \in \mathbb{N}\}$.

We define a map $\varphi : X \to \Psi(\mathcal{A})$ as follows: $\varphi(\mathbb{C} \times \{1/n\}) = \{n\}$, and $\varphi((r,0)) = A_r$. It is a routine to check that φ is a continuous onto, 1-sequence-covering, at most boundary-one and s-map. Additionally φ is even open. □

Let $f : X \to Y$ be a continuous onto map. Then f is said to be *1-scc* if for each compact set $K \subset Y$, there is a compact set $L \subset X$ such that: $f(L) = K$, and for each $y \in K$ we can take a point $x_y \in L$ such that if $y_n \in Y$ and $y_n \to y$, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset X$ with $x_n \to x_y$ and $x_n \in f^{-1}(y_n)$; f is said to be *scc* if for each compact set $K \subset Y$, there is a compact set $L \subset X$ such that: $f(L) = K$, and for each sequence $\{y_n : n \in \mathbb{N}\} \subset Y$ converging to a point in K, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset X$ converging to a point in L with $x_n \in f^{-1}(y_n)$. A 1-*scc* map is *scc*.

Question 2.9 ([4, Question 2.7.16])**.** Is every *scc*-map of a compact space 1-*scc*?

This question is in the negative.

Theorem 2.10. *Not every scc-map of a compact space is 1-scc.*

Proof. Recall the sequential fan $S_\omega = \{\infty\} \cup \{(n,m) : n, m \in \mathbb{N}\}$ and take its subspaces $A_n = \{\infty\} \cup \{(k,m) : 1 \le k \le n,\ m \in \mathbb{N}\}$. Let X be the topological sum $\oplus\{A_n : n \in \mathbb{N}\}$ and let $\varphi : X \to S_\omega$ be the map such that $\varphi \restriction A_n$ is the natural embedding. We consider the extension $\varphi^\beta : \beta X \to \beta S_\omega$ of φ to the Stone-Čech compactifications. We see that φ^β is an *scc*-map which is not 1-*scc*. Since φ^β is a map of a compact space, to show that φ^β is *scc*, it is enough to show that φ^β is sequence-covering. Since a 1-*scc*-map is 1-sequence-covering, to show that φ^β is not 1-*scc*, it is enough to show that φ^β is not 1-sequence-covering.

Claim 1: Let $\{p_n : n \in \mathbb{N}\} \subset \beta S_\omega$ be a sequence converging $p \in \beta S_\omega \setminus \{p_n : n \in \mathbb{N}\}$. Then, $p = \infty$ and $p_n \in S_\omega$ for all but finitely many $n \in \mathbb{N}$.

Proof. Assume $p \in \beta S_\omega \setminus S_\omega$. Take an $f \in \mathbb{N}^\mathbb{N}$ such that $p \notin \overline{N(f)}^{\beta S_\omega}$. Then, the open set $\overline{S_\omega \setminus N(f)}^{\beta S_\omega}$ contains p and p_n for all but finitely many $n \in \mathbb{N}$. Since $\overline{S_\omega \setminus N(f)}^{\beta S_\omega}$ is homeomorphic to $\beta\mathbb{N}$, this is a contradiction. Thus we have $p = \infty$. Assume that $\beta S_\omega \setminus S_\omega$ contains a subsequence $\{p_{k_n} : n \in \mathbb{N}\}$ of $\{p_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, take an $f_n \in \mathbb{N}^\mathbb{N}$ such that $p_{k_n} \notin \overline{N(f_n)}^{\beta S_\omega}$. Take an $f \in \mathbb{N}^\mathbb{N}$ with $f_n \le^* f$ [1]. Then $\overline{N(f)}^{\beta S_\omega} \setminus N(f) \subset \overline{N(f_n)}^{\beta S_\omega}$. Hence, $\overline{N(f)}^{\beta S_\omega} \cap \{p_{k_n} : n \in \mathbb{N}\} = \emptyset$. Since $\overline{N(f)}^{\beta S_\omega}$ is open in βS_ω, this is a contradiction. □

Since φ is sequence-covering, so is φ^β by Claim 1.

Claim 2: Let $\{p_n : n \in \mathbb{N}\} \subset \beta X$ be a sequence converging $p \in \beta X \setminus \{p_n : n \in \mathbb{N}\}$. Then, p is a limit point in X and $p_n \in X$ for all but finitely many $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ and $f \in \mathbb{N}^\mathbb{N}$, let

$$A_n(f) = \{\infty\} \cup \{(k,m) : 1 \le k \le n,\ m \ge f(n)\} \text{ and } X(f) = \oplus\{A_n(f) : n \in \mathbb{N}\}.$$

Let L be the set of all limit points in X. Since $\overline{L}^{\beta X}$ is homeomorphic to $\beta\mathbb{N}$, it contains only finitely many p_n's. We may assume $\overline{L}^{\beta X} \cap \{p_n : n \in \mathbb{N}\} = \emptyset$. Assume $p \in \beta X \setminus X$. We consider the case that X contains a subsequence $\{p_{k_n} : n \in \mathbb{N}\}$ of $\{p_n : n \in \mathbb{N}\}$. Then $\{p_{k_n} : n \in \mathbb{N}\} \cap A_n$ must be finite for all $n \in \mathbb{N}$, so $\overline{\{p_{k_n} : n \in \mathbb{N}\}}^{\beta X}$ is homeomorphic to $\beta\mathbb{N}$. This is a contradiction. Hence X contains only finitely many p_n's. For simplicity, we may assume $p_n \in \beta X \setminus X$ for all $n \in \mathbb{N}$. Using the condition $p_n \notin \overline{L}^{\beta X}$, we can take an $f_n \in \mathbb{N}^\mathbb{N}$ such that $p_n \notin \overline{X(f_n)}^{\beta X}$. Take an $f \in \mathbb{N}^\mathbb{N}$ with $f_n \le^* f$ for all $n \in \mathbb{N}$. Then

[1] $f_n \le^* f$ stands for $f_n(k) \le f(k)$ for all but finitely many $k \in \mathbb{N}$.

$\overline{X(f)}^{\beta X} \setminus X(f) \subset \overline{X(f_n)}^{\beta X}$. Hence $\overline{X(f)}^{\beta X} \cap \{p_n : n \in \mathbb{N}\} = \emptyset$. In other words, $\{p_n : n \in \mathbb{N}\} \subset \overline{X \setminus X(f)}^{\beta X}$. Since$\overline{X \setminus X(f)}^{\beta X}$ is homeomorphic to $\beta\mathbb{N}$, this is a contradiction. Consequently we have $p \in L$. Since X is open in βX, it contains all p_n but finitely many $n \in \mathbb{N}$. □

By Claim 2 and the fact that φ is not 1-sequence-covering, φ^β is not 1-sequence-covering at $\infty \in \beta S_\omega$. □

For a weak-base $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ of a space X and $A \subset X$, the family $\bigcup\{\mathcal{P}_x : x \in A\}$ is said to be an *outer weak-base* of A.

Question 2.11 ([4, Question 2.7.20])**.** Does every compact subset of a space with a point-countable weak-base have a countable outer weak-base?

This question is in the affirmative.

Proposition 2.12. *Let X be a space with a point-countable weak-base. Then every compact subset of X has a countable outer weak-base.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a point-countable weak-base for X, and let K be a compact subset in X. Since every point-countable weak-base is a k-network [9, Proposition 1.8.(1)], K is metrizable. Let D be a countable dense subset in K. Since $\mathcal{P}$ is point-countable, the family $\mathcal{Q} = \{P \in \mathcal{P} : P \cap D \neq \emptyset\}$ is countable. If $x \in K$ and $P \in \mathcal{P}_x$, then $P \in \mathcal{Q}$, because there is a sequence in D converging to x and P is a sequential neighborhood at x. Thus $\bigcup\{\mathcal{P}_x : x \in K\}$ is countable. □

A boundary-compact sequence-covering map of a first-countable space (in particular, a metric space) is 1-sequence-covering [4, Theorem 3.5.3]. A space X is said to be *g-second countable* (resp., *g-metrizable*) if it has a countable (resp., σ-locally finite) weak-base. A g-second countable space is g-metrizable.

Question 2.13 ([4, Question 3.5.8])**.** Let $f : X \to Y$ be a boundary-compact sequence-covering map. If X is g-metrizable, is f 1-sequence-covering?

This question is in the negative.

Proposition 2.14. *There is a boundary-compact, sequence-covering map $\varphi : X \to Y$ which is not 1-sequence-covering such that X is g-second countable.*

Proof. For each $n \in \mathbb{N}$, let $A_n = \{\infty\} \cup \{(k, m) : 1 \leq k \leq n, m \in \mathbb{N}\}$ be the subspace of the sequential fan $S_\omega = \{\infty\} \cup \{(n, m) : n, m \in \mathbb{N}\}$. Consider the topological sum $(\oplus\{A_n : n \in \mathbb{N}\}) \oplus \mathbb{S}$, and let X be the quotient space of

$(\oplus\{A_n : n \in \mathbb{N}\}) \oplus \mathbb{S}$ obtained by identifying the point $\infty \in A_n$ and $1/n \in \mathbb{S}$ for each $n \in \mathbb{N}$. It is a routine to check that X is g-second countable. Let $\varphi : X \to S_\omega$ be the map defined as follows: $\varphi(x) = \infty$ for $x \in \mathbb{S}$, and $\varphi((k,m)) = (k,m)$. Easily we can see that φ is continuous onto, sequence-covering, boundary-compact and not 1-sequence-covering. □

Let $f : X \to Y$ be a sequence-covering closed map and assume that X has a point-countable weak-base, then Y is gf-countable [6].

Question 2.15 ([4, Question 3.5.19])**.** Let $f : X \to Y$ be a sequence-covering closed map and assume that X has a point-countable sn-network. Is Y snf-countable?

Let $f : X \to Y$ be a sequence-covering closed map and assume that X has a σ-compact-finite weak-base, then Y also has a σ-compact-finite weak-base [6].

Question 2.16 ([4, Question 4.1.29])**.** Let $f : X \to Y$ be a sequence-covering closed map and assume that X has a σ-compact-finite sn-network. Does Y have a σ-compact-finite sn-network?

These two questions are in the negative. Note that a σ-compact-finite family is point-countable, and that a space with a σ-compact-finite sn-network is snf-countable. For convenience of the readers, we give the proof of the following well known fact.

Lemma 2.17. *The sequential fan S_ω is not snf-countable at the point ∞.*

Proof. Assume that the point ∞ has a countable sn-network $\{A_n : n \in \mathbb{N}\}$. Then, $N(f_n) \subset A_n$ for some $f_n \in \mathbb{N}^{\mathbb{N}}$. Hence S_ω is first-countable at ∞. This is a contradiction. □

Theorem 2.18. *There is a sequence-covering closed map $\varphi : Y \to S_\omega$ such that Y has a σ-compact-finite sn-network.*

Proof. Let $\mathbb{N} \subset X \subset \beta\mathbb{N}$ be a countably compact space such that every compact subset of X is finite. Such a space was constructed by Frolik [1]. For each $k, l \in \mathbb{N}$ and a function $f \in \mathbb{N}^{\mathbb{N}}$, we put

$$A_k = \{(n,m,k) : n,m \in \mathbb{N}, n \le k\},$$
$$A_k(l) = \{(n,m,k) \in A_k : m \ge l\}, \text{ and}$$
$$A_k(f) = \{(n,m,k) \in A_k : m \ge f(n)\}.$$

Let $Y = X \cup \bigcup\{A_k : k \in \mathbb{N}\}$. We give Y a topology. Every point in $\bigcup\{A_k : k \in \mathbb{N}\}$ is isolated in Y. Every point $k \in \mathbb{N}$ in Y has a basic open neighborhood of the form $\{k\} \cup A_k(l)$, $l \in \mathbb{N}$. Every point $p \in Y \setminus (\mathbb{N} \cup \bigcup\{A_k : k \in \mathbb{N}\})$ has a basic open neighborhood of the form, for an open neighborhood U of p in X and an $f \in \mathbb{N}^{\mathbb{N}}$, $U(p, f) = U \cup \bigcup\{A_k(f) : k \in U\}$. Obviously Y is a zero-dimensional Tychonoff space, and each $\{k\} \cup A_k$ is homeomorphic to the convergent sequence $\mathbb{S}$.

Claim: Let $p \in Y \setminus (\mathbb{N} \cup \bigcup\{A_k : k \in \mathbb{N}\})$, and $\{y_n : n \in \mathbb{N}\}$ be a sequence in Y converging to p. Then $\{y_n : n \in \mathbb{N}\}$ is finite.

Proof. We may assume $\{y_n : n \in \mathbb{N}\} \subset \bigcup\{A_k : k \in \mathbb{N}\}$, because every compact subset of X is finite. Let $P = \{k \in \mathbb{N} : A_k \cap \{y_n : n \in \mathbb{N}\} \neq \emptyset\}$. If $P \notin p$, $\mathbb{N} \setminus P \in p$, so there is an open neighborhood U of p in X such that $U \cap P = \emptyset$. Then $\{y_n : n \in \mathbb{N}\} \cap U(p, c_1) = \emptyset$, where c_1 is the constant function to 0. This is a contradiction. Let $P \in p$. Let $\{P_0, P_1\}$ be a partition of P of infinite sets, and assume $P_1 \in p$. Then by a similar argument as in the case $P \notin p$, we can observe that $\{y_n : n \in P_0\}$ does not converge to p. Thus $\{y_n : n \in \mathbb{N}\}$ cannot be a convergent sequence to p. $\square$

We see that Y has a σ-compact-finite sn-network. For each $y \in Y \setminus \mathbb{N}$, let $\mathcal{P}_y = \{\{y\}\}$, and for each $y = k \in \mathbb{N}$ in Y, let $\mathcal{P}_y = \{\{k\} \cup A_k(l) : l \in \mathbb{N}\}$. By Claim above, each $\mathcal{P}_y$ is an sn-network at y. Next we see that the family $\bigcup\{\mathcal{P}_y : y \in Y\}$ is σ-compact-finite. We put

$\mathcal{Q} = \{\{p\} : p \in Y \setminus (\mathbb{N} \cup \bigcup\{A_k : k \in \mathbb{N}\})\}$,

$\mathcal{Q}_{n,m,k} = \{(n, m, k)\}$ for each $n, m, k \in \mathbb{N}$ with $n \leq k$, and

$\mathcal{R}_{k,l} = \{\{k\} \cup A_k(l)\}$ for each $k, l \in \mathbb{N}$.

Obviously these are compact-finite, and the union of them is just $\bigcup\{\mathcal{P}_y : y \in Y\}$.

We define a map $\varphi : Y \to S_\omega$ as follows: $\varphi(y) = \infty$ if $y \in X$, and $\varphi((n, m, k)) = (n, m)$ for each $n, m, k \in \mathbb{N}$ with $n \leq k$. For each $f \in \mathbb{N}^{\mathbb{N}}$, we can easily observe $\varphi^{-1}(N(f)) = X \cup \bigcup\{A_k(f) : k \in \mathbb{N}\}$, so this map is continuous. Obviously φ is sequence-covering. Finally we see that φ is closed. Let H be a closed subset of Y. If $H \cap X \neq \emptyset$, then obviously $\varphi(H)$ is closed in S_ω. Assume that $H \cap X = \emptyset$ and $\varphi(H)$ is not closed in S_ω. Then there are an $n_1 \in \mathbb{N}$ and a sequence $m_1 < m_2 < \cdots$ such that $\{(n_1, m_j) : j \in \mathbb{N}\} \subset \varphi(H)$. Take $k_j \in \mathbb{N}$ such that $(n_1, m_j, k_j) \in H$ and $n_1 \leq k_j$. If $\{k_j : j \in \mathbb{N}\}$ is finite, H contains a sequence converging to some point $k \in \mathbb{N}$. This is a contradiction. So, without loss of generality, we may assume $k_1 < k_2 < \cdots$. Since X is countably compact, there is a point $p \in \overline{\{k_j : j \in \mathbb{N}\}} \setminus \{k_j : j \in \mathbb{N}\}$. Since H is closed in Y, there are an open neighborhood U of p in X and an $f \in \mathbb{N}^{\mathbb{N}}$ such that $U(p, f) \cap H = \emptyset$. Since U

contains infinitely many k_j, there is a $j \in \mathbb{N}$ such that $k_j \in U$ and $f(n_1) \leq m_j$. For this j, $(n_1, m_j, k_j) \in A_{k_j}(f) \cap H \subset U(p, f) \cap H$. This is a contradiction. Thus $\varphi(H)$ must be closed in S_ω. □

Question 2.19 ([4, Question 3.5.27])**.** Give a characterization of a space X such that every pseudo-sequence-covering map onto X is 1-sequence-covering.

The following answers this question.

Proposition 2.20. *For a space X, the following are equivalent.*

(1) *Every pseudo-sequence-covering map onto X is 1-sequence-covering;*
(2) *Every pseudo-sequence-covering map onto X is sequence-covering;*
(3) *Every convergent sequence of X is a finite set:*
(4) *Every map onto X is 1-sequence-covering.*

PROOF. Among implications (1) → (2) → (3) → (4) → (1), we have only to show (2) → (3). Assume (2). Since X is regular T_1, there are a regular T_1 extremally disconnected space EX and a perfect irreducible onto map $f : EX \to X$ [10], where a space is said to be extremally disconnected if the closure of each open subset is open. Since a perfect map is pseudo-sequence-covering, f must be sequence-covering by (2). Hence (3) holds, because every convergent sequence in an extremally disconnected space must be a finite set. □

A family $\mathcal{P}$ of subsets of a space X is said to be *hereditarily closure-preserving* if for any $\mathcal{P}' \subset \mathcal{P}$ and any subsets $H(P) \subset P$ for $P \in \mathcal{P}'$,

$$\overline{\bigcup\{H(P) : P \in \mathcal{P}'\}} = \bigcup\{\overline{H(P)} : P \in \mathcal{P}'\}$$

holds. In a regular T_1-space X, if $\mathcal{P}$ is a hereditarily closure-preserving family of X, then so is $\overline{\mathcal{P}} = \{\overline{P} : P \in \mathcal{P}\}$.

Question 2.21 ([4, Question 4.2.7])**.** Let $\mathcal{P}$ be a hereditarily closure-preserving family of a T_2 (non-regular) space X. Is $\overline{\mathcal{P}} = \{\overline{P} : P \in \mathcal{P}\}$ hereditarily closure-preserving?

This question is in the negative.

Proposition 2.22. *There are a T_2 (non-regular) space X and a hereditarily closure-preserving family $\mathcal{P}$ in X such that $\overline{\mathcal{P}} = \{\overline{P} : P \in \mathcal{P}\}$ is not hereditarily closure-preserving.*

PROOF. For each $n, m \in \mathbb{N}$, let $A_{n,m} = \{(n, m)\} \cup \{(n, m, l) : l \in \mathbb{N}\}$, and we put $X = \{\infty\} \cup \bigcup\{A_{n,m} : n, m \in \mathbb{N}\}$. We give X a topology. Each (n, m, l) is isolated in X. Each (n,m) has a basic open neighborhood of the form $A_{n,m}(k) = \{(n,m)\} \cup \{(n,m,l) : l \geq k\}$, $k \in \mathbb{N}$. Thus $A_{n,m}$ is homeomorphic to $\mathbb{S}$. The point ∞ has a basic open neighborhood of the form, for $k \in \mathbb{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$, $V(\infty, k, f) = \{\infty\} \cup \{(n,m) : n \in \mathbb{N}, n \geq k, m \geq f(n)\} \cup \{(n,m,l) : n, l \in \mathbb{N}, m \geq f(n)\}$. Note that $V(\infty,k,f) \cap V(\infty,k',f') = V(\infty, \max\{k,k'\}, \max\{f,f'\})$. Obviously X with this topology is a T_2 non-regular space (for example, the closed set $\{(1,m) : m \in \mathbb{N}\}$ and the point ∞ cannot be separated by open sets). For each $n \in \mathbb{N}$, let $P_n = \{(n,m,l) : m, l \in \mathbb{N}\}$, and we put $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$.

We see that $\mathcal{P}$ is hereditarily closure-preserving in X. Fix a subset $J \subset \mathbb{N}$, and take any subset $Q_j \subset P_j$ for each $j \in J$. We observe that $\bigcup\{\overline{Q}_j : j \in J\}$ is closed in X. Let $x \in X \setminus \bigcup\{\overline{Q}_j : j \in J\}$. If $x = (n,m)$ for some $n, m \in \mathbb{N}$, $\{(n,m,l) : l \in \mathbb{N}\} \cap \bigcup\{Q_j : j \in J\}$ is finite. hence $A_{n,m}(k) \cap \bigcup\{Q_j : j \in J\} = \emptyset$ for some $k \in \mathbb{N}$. Let $x = \infty$. By the condition $\infty \notin \overline{Q}_j$, there is a $k_j \in \mathbb{N}$ such that $Q_j \cap \bigcup\{A_{j,m} : m \geq k_j\} = \emptyset$. Let $f \in \mathbb{N}^{\mathbb{N}}$ be any function with $f(j) = k_j$ for $j \in J$. Then $V(\infty, 1, f) \cap \bigcup\{Q_j : j \in J\} = \emptyset$. Thus $\bigcup\{\overline{Q}_j : j \in J\}$ is closed in X.

Finally we see that $\overline{\mathcal{P}} = \{\overline{P}_n : n \in \mathbb{N}\}$ is not hereditarily closure-preserving. For each $n \in \mathbb{N}$, let $C_n = \{(n,m) : m \in \mathbb{N}\}$. Then each C_n is closed in X and $C_n \subset \overline{P}_n$, but easily we can see $\infty \in \overline{\bigcup\{C_n : n \in \mathbb{N}\}}$. □

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