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Characterizing s-paratopological groups by free paratopological groups ☆

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ABSTRACT

A paratopological group G is called an *s-paratopological group* if every sequentially continuous homomorphism from G to a paratopological group is continuous. In this paper, the structure of s-paratopological groups is established in terms of free paratopological groups. Namely, if G is a non-discrete T_1 paratopological group, then the following statements (1), (2), (3) and (4) are equivalent. (1) G is an s-paratopological group. (2) G is topologically isomorphic to a quotient group of a free paratopological group on a metrizable space. (3) G is topologically isomorphic to a quotient group of a free paratopological group on a T_1 Fréchet space. (4) G is topologically isomorphic to a quotient group of a free paratopological group on a T_1 sequential space.

1. Introduction

A topological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous and the inverse mapping of G on itself is also continuous. The important notion of an s-topological group was introduced by N. Noble [11]. A topological group G is called an *s-topological group* if every sequentially continuous homomorphism from G to a topological group is continuous. Recall that a mapping $f : X \to Y$ between topological spaces X and Y is said to be *sequentially continuous* if

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$\{f(x_n)\}_{n\in\omega}$ converges to $f(x)$ in Y whenever a sequence $\{x_n\}_{n\in\omega}$ converges to x in X. In the past, some topologists were interested in investigating the properties of s-topological groups [2,8,9,11,18] etc.

A paratopological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. The absence of the continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups. Concerning a recent survey in the theory of paratopological groups, readers may consult [20]. Many publications have appeared in this field with regard to one important question, i.e., when are various results on topological groups valid for paratopological groups? As a generalization of free topological groups, S. Romaguera, M. Sanchis, and M. Tkachenko [17] introduced free paratopological groups on arbitrary topological spaces and discussed some of their topological properties. M. Tkachenko [20, p. 851] thought that free paratopological groups should be a very useful tool for the study of general paratopological groups.

Definition 1.1. [17] Let X be a subspace of a paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X\rangle = G$; and
(2) every continuous mapping $f : X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group* (briefly, *free paratopological group*) on X and is denoted by $FP(X)$.

Let $F_a(X)$ denote the algebraic free group on a non-empty set X and e be the identity of $F_a(X)$. The set X is called the free basis of $F_a(X)$. Here are some details, for instance, see [3,16]. Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $x_1, ..., x_n \in X$ and $\epsilon_1, ..., \epsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say in this case that the length $l(g)$ of g equals to n. Every element $g \in F_a(X)$ distinct from the identity e can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$ and $x_i \neq x_{i+1}$ for every $i = 1, ..., n-1$.

Remark 1.2. It is well known that the topology of $FP(X)$ is the finest paratopological group topology on the abstract free group $F_a(X)$ which induces the original topology on X [17].

Remark 1.3. If X is a T_1-space, then $FP(X)$ is also T_1 and X^{-1} is a closed and discrete subspace of $FP(X)$ [5].

In this paper, inspired by the concept of an s-topological group, we shall introduce s-paratopological groups and make the first step towards the study of them. Our main purpose is to characterize s-paratopological groups in terms of free paratopological groups, hence, to establish the structure of s-paratopological groups.

Definition 1.4. A paratopological group G is called an *s-paratopological group* if every sequentially continuous homomorphism from G to a paratopological group is continuous.

Let X be a topological space. A subset P of X is called a *sequential neighborhood* of $x \in X$ in X if any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x is eventually in P, i.e., $\{x_n : n \geq k_0\} \cup \{x\} \subset P$ for some $k_0 \in \mathbb{N}$. P is called a *sequentially open* subset of X if P is a sequential neighborhood of every point of P in X. The space X is called a *sequential space* [7] if every sequentially open subset of X is open in X. It is not difficult to check that every sequential paratopological group is an s-paratopological group. Particularly, the Sorgenfrey line is an s-paratopological group. By [19, Corollary 2.5], it is easy to see that there exists a topological group which is not an s-paratopological group. For example, the free topological group on $\beta\mathbb{N}$ is such an example.

- The symbol $\mathbb{N}$ denotes the set of all positive integers and $\omega = \{0\} \cup \mathbb{N}$. For every $n \in \mathbb{N}$, $\mathbb{S}_n$ denotes the group of all permutations on the set $\{0, 1, ..., n-1\}$.
- Let G be a group and $\{A_m : m \leq n\}$ be a family of non-empty subsets of G for $n \in \mathbb{N}$. $A_1 \cdots A_n$ denotes the set $\{a_1 \cdots a_n : a_m \in A_m, m \leq n\}$.
- Let G be a group and $\{A_n\}_{n\in\omega}$ be a sequence of non-empty subsets of G. Following [12, Definition 3.1.3], we write

$$SP_{m\leq n} A_m = \bigcup_{\sigma\in\mathbb{S}_{n+1}} A_{\sigma(0)} A_{\sigma(1)} \cdots A_{\sigma(n)}$$

and

$$SP_{n\in\omega} A_n = \bigcup_{n\in\omega} SP_{m\leq n} A_m = \bigcup_{n\in\omega} \bigcup_{\sigma\in\mathbb{S}_{n+1}} A_{\sigma(0)} A_{\sigma(1)} \cdots A_{\sigma(n)}.$$

For some unexplained terminology, readers may consult [3,6].

2. s-Paratopological groups and paratopologized sets

In this section, we introduce a crucial technical definition named *paratopologized set* to establish a few theorems required in order to describe the structure of s-paratopological groups in terms of free paratopological groups.

The following Lemma immediately follows from [12, Lemma 3.1.1].

Lemma 2.1. *Let U be a neighborhood of the identity e in a paratopological group G. Then there exists a sequence $\{V_n\}_{n\in\omega}$ consisting of neighborhoods of the identity e in G such that $SP_{n\in\omega} V_n \subset U$.*

The following description of a neighborhood base at the identity of a paratopological group is well known. For example, it appears in [10,14,15,17] etc.

Lemma 2.2. *Let G be a paratopological group and $\mathcal{N}$ be a base at the identity e of G. Then the family $\mathcal{N}$ has the following four properties.*

(1) *for every $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ with $W \subset U \cap V$;*
(2) *for every $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $VV \subset U$;*
(3) *for every $U \in \mathcal{N}$ and $g \in U$, there exists $V \in \mathcal{N}$ such that $gV \subset U$;*
(4) *for every $U \in \mathcal{N}$ and $g \in G$, there exists $V \in \mathcal{N}$ such that $gVg^{-1} \subset U$.*

Conversely, if $\mathcal{N}$ is a family of subsets of an abstract group G containing the identity e of G and satisfying (1)–(4), then G admits the unique topology $\mathcal{T}$ that makes it a paratopological group with $\mathcal{N}$ being a base at e. In addition, if $\{e\} = \bigcap \mathcal{N}$, then the topology $\mathcal{T}$ satisfies the T_1-separation axiom.

Definition 2.3. Let G be an abstract group and $\mathcal{S}$ be a set of sequences in G. The set $\mathcal{S}$ is called a *paratopologized set* (briefly, *PT-set*) in G if there is a T_1 paratopological group topology on G in which all sequences of $\mathcal{S}$ converge to the identity e of G. The finest T_1 paratopological group topology on G with this property is denoted by $\tau_{\mathcal{S}}$.

Here, we give a criterion for a set to be a PT-set in an abstract group.

Theorem 2.4. *Let G be a group with identity e and $\mathcal{S} = \{S_i : i \in I\}$ be a set of sequences in G, where $S_i = \{x_n^i\}_{n\in\omega}$ for every $i \in I$. Then the following statements (a), (b), and (c) are equivalent.*

(a) *The topology $\tau_{\mathcal{S}}$ on G exists;*
(b) *$\mathcal{S}$ is a PT-set in G;*
(c) *$\bigcap_{f\in\mathcal{F}} SP_{n\in\omega}A_n(f) = \{e\}$, where $\mathcal{F}$ denotes the set of all mappings f from $\omega \times I \times G$ to ω such that $f(k,i,g) < f(k+1,i,g)$ for arbitrary $k \in \omega, i \in I$ and $g \in G$;*

$$A_n(f) = \bigcup_{i\in I}\bigcup_{g\in G} g^{-1}A^i_{f(n,i,g)}g$$

for every $n \in \omega$; and

$$A^i_m = \{e\} \cup \{x^i_n : n \geq m\}$$

for every $m \in \omega$ and $i \in I$.

Moreover, if one of the statements (a), (b) or (c) holds, then the family $\{SP_{n\in\omega}A_n(f) : f \in \mathcal{F}\}$ is a base at the identity e in $(G, \tau_{\mathcal{S}})$.

Proof. (a) ⇒ (b). This is obvious by Definition 2.3.

(b) ⇒ (c). The set $\mathcal{S}$ being a PT-set in G, there is a T_1 paratopological group topology τ' on G in which all sequences of $\mathcal{S}$ converge to the identity e. It suffices to prove that for every open neighborhood U of e in (G, τ'), there exists a mapping $f \in \mathcal{F}$ such that $SP_{n\in\omega}A_n(f) \subset U$. At first, by Lemma 2.1, there exists a sequence $\{V_n\}_{n\in\omega}$ consisting of neighborhoods of e in (G, τ') such that $SP_{n\in\omega}V_n \subset U$. Let $k \in \omega$ be arbitrary. For every $i \in I$ and $g \in G$, since the sequence S_i converges to e in (G, τ'), we may choose $f(k,i,g)$ such that $g^{-1}x^i_n g \in V_k$ when $n \geq f(k,i,g)$, whence $A_k(f) \subset V_k$. Without loss of generality, we may assume that $f(k,i,g) < f(k+1,i,g)$ for any $k \in \omega, i \in I$, and $g \in G$. Thus $f \in \mathcal{F}$ and $SP_{n\in\omega}A_n(f) \subset SP_{n\in\omega}V_n \subset U$.

(c) ⇒ (a).

Claim. *The family $\mathcal{N} = \{SP_{n\in\omega}A_n(f) : f \in \mathcal{F}\}$ is a base at the identity e for some T_1 paratopological group topology σ on G.*

In order to prove the above claim, we have to check that $\mathcal{N}$ satisfies conditions (1), (2), (3) and (4) in Lemma 2.2.

(1) Let $SP_{n\in\omega}A_n(f_1), SP_{n\in\omega}A_n(f_2) \in \mathcal{N}$. For every $k \in \omega, i \in I$ and $g \in G$, let

$$f(k,i,g) = f_1(k,i,g) + f_2(k,i,g).$$

Then $f \in \mathcal{F}$,

$$A^i_{f(k,i,g)} \subset A^i_{f_1(k,i,g)} \cap A^i_{f_2(k,i,g)}$$

and so

$$A_k(f) \subset A_k(f_1) \cap A_k(f_1).$$

Hence,

$$SP_{n\in\omega}A_n(f) \subset SP_{n\in\omega}A_n(f_1) \cap SP_{n\in\omega}A_n(f_2).$$

(2) Let $SP_{n\in\omega}A_n(f) \in \mathcal{N}$. For every $k \in \omega, i \in I$ and $g \in G$, let

$$\psi(k, i, g) = f(2k + 1, i, g).$$

Obviously, $\psi \in \mathcal{F}$ and $A_k(\psi) = A_{2k+1}(f)$ for every $k \in \omega$. Let $k, l \in \omega$. Suppose $\alpha \in \mathbb{S}_{k+1}$ and $\beta \in \mathbb{S}_{l+1}$.

Case 1. Assume that $k \leq l$. Put

$$\begin{aligned}
&\sigma(r) = 2\alpha(r) + 1, \text{ if } 0 \leq r \leq k;\\
&\sigma(k + 1 + q) = 2\beta(q), \text{ if } 0 \leq \beta(q) \leq k \text{ and } 0 \leq q \leq l;\\
&\sigma(k + 1 + q) = k + 1 + \beta(q), \text{ if } k < \beta(q) \leq l \text{ and } 0 \leq q \leq l.
\end{aligned}$$

Then $\sigma \in \mathbb{S}_{k+l+2}$. Since $A_{m+1}(f) \subset A_m(f)$ for every $m \in \omega$, we have

$$\begin{aligned}
&A_{\alpha(r)}(\psi) = A_{2\alpha(r)+1}(f) = A_{\sigma(r)}(f), \text{ if } 0 \leq r \leq k;\\
&A_{\beta(q)}(\psi) = A_{2\beta(q)+1}(f) \subset A_{2\beta(q)}(f) = A_{\sigma(k+1+q)}(f), \text{ if } 0 \leq \beta(q) \leq k \text{ and } 0 \leq q \leq l;\\
&A_{\beta(q)}(\psi) = A_{2\beta(q)+1}(f) \subset A_{k+1+\beta(q)}(f) = A_{\sigma(k+1+q)}(f), \text{ if } k < \beta(q) \leq l \text{ and } 0 \leq q \leq l.
\end{aligned}$$

Hence,

$$A_{\alpha(0)}(\psi) \cdots A_{\alpha(k)}(\psi) A_{\beta(0)}(\psi) \cdots A_{\beta(l)}(\psi) \subset A_{\sigma(0)}(f) \cdots A_{\sigma(k+l+1)}(f) \subset SP_{n\in\omega}A_n(f).$$

Thus,

$$SP_{n\in\omega}A_n(\psi) SP_{n\in\omega}A_n(\psi) \subset SP_{n\in\omega}A_n(f).$$

Case 2. Assume that $k > l$. Put

$$\begin{aligned}
&\sigma(r) = 2\alpha(r) + 1, \text{ if } 0 \leq \alpha(r) \leq l \text{ and } 0 \leq r \leq k;\\
&\sigma(r) = \alpha(r) + l + 1, \text{ if } l < \alpha(r) \leq k \text{ and } 0 \leq r \leq k;\\
&\sigma(k + 1 + q) = 2\beta(q), \text{ if } 0 \leq q \leq l.
\end{aligned}$$

Then $\sigma \in \mathbb{S}_{k+l+2}$. Since $A_{m+1}(f) \subset A_m(f)$ for every $m \in \omega$, we have

$$\begin{aligned}
&A_{\alpha(r)}(\psi) = A_{2\alpha(r)+1}(f) = A_{\sigma(r)}(f), \text{ if } 0 \leq \alpha(r) \leq l \text{ and } 0 \leq r \leq k;\\
&A_{\alpha(r)}(\psi) = A_{2\alpha(r)+1}(f) \subset A_{\alpha(r)+l+1}(f) = A_{\sigma(r)}(f), \text{ if } l < \alpha(r) \leq k \text{ and } 0 \leq r \leq k;\\
&A_{\beta(q)}(\psi) = A_{2\beta(q)+1}(f) \subset A_{2\beta(q)}(f) = A_{\sigma(k+1+q)}(f), \text{ if } 0 \leq q \leq l.
\end{aligned}$$

Hence,

$$A_{\alpha(0)}(\psi) \cdots A_{\alpha(k)}(\psi) A_{\beta(0)}(\psi) \cdots A_{\beta(l)}(\psi) \subset A_{\sigma(0)}(f) \cdots A_{\sigma(k+l+1)}(f) \subset SP_{n\in\omega}A_n(f).$$

Thus,

$$SP_{n\in\omega}A_n(\psi) SP_{n\in\omega}A_n(\psi) \subset SP_{n\in\omega}A_n(f).$$

(3) Let $SP_{n\in\omega}A_n(f) \in \mathcal{N}$ and $x \in SP_{n\in\omega}A_n(f)$. Let

$$k = \min\{n \in \omega : x \in \bigcup_{\sigma\in\mathbb{S}_{n+1}} A_{\sigma(0)}(f)A_{\sigma(1)}(f)\cdots A_{\sigma(n)}(f)\}.$$

Then $x \in A_{\alpha(0)}(f)\cdots A_{\alpha(k)}(f)$ for some $\alpha \in \mathbb{S}_{k+1}$. For every $k \in \omega, i \in I$ and $g \in G$, put

$$\phi(n, i, g) = f(k + 1 + n, i, g).$$

Clearly, $\phi \in \mathcal{F}$. Let $l \in \omega$ and $\beta \in \mathbb{S}_{l+1}$ be arbitrary. Put

$$\sigma(r) = \alpha(r), \text{ if } 0 \le r \le k;$$

$$\sigma(k + 1 + r) = k + 1 + \beta(r), \text{ if } 0 \le r \le l.$$

Then $\sigma \in \mathbb{S}_{k+l+2}$. For every $0 \le r \le l$, we have

$$A_{\beta(r)}(\phi) = A_{k+1+\beta(r)}(f) = A_{\sigma(k+1+r)}(f).$$

So

$$xA_{\beta(0)}(\phi)\cdots A_{\beta(l)}(\phi) \subset A_{\sigma(0)}(f)\cdots A_{\sigma(k+l+1)}(f) \subset SP_{n\in\omega}A_n(f).$$

Hence

$$xSP_{n\in\omega}A_n(\phi) \subset SP_{n\in\omega}A_n(f).$$

(4) Let $SP_{n\in\omega}A_n(f) \in \mathcal{N}$ and $h \in G$. For every $k \in \omega, i \in I$ and $g \in G$, let

$$\varphi(k, i, g) = f(k, i, gh).$$

Obviously, $\varphi \in \mathcal{F}$. For every $k \in \omega$ and $i \in I$, we have

$$\begin{aligned}\bigcup_{g\in G} g^{-1}A^i_{\varphi(k,i,g)}g &= \bigcup_{g\in G} g^{-1}A^i_{f(k,i,gh)}g \\ &= \bigcup_{g\in G} h(gh)^{-1}A^i_{f(k,i,gh)}(gh)h^{-1} \\ &\subset h(\bigcup_{g\in G} g^{-1}A^i_{f(k,i,g)}g)h^{-1}.\end{aligned}$$

So $A_k(\varphi) \subset hA_k(f)h^{-1}$, i.e., $h^{-1}A_k(\varphi)h \subset A_k(f)$ for every $k \in \omega$, whence

$$h^{-1}SP_{n\in\omega}A_n(\varphi)h \subset SP_{n\in\omega}A_n(f).$$

We have completed the proof of the above claim. Now, the topology σ on G is finer than an arbitrary paratopological group topology τ in which every sequence of $\mathcal{S}$ converges to e in (G, τ). Indeed, suppose $O \in \tau$. For every $g \in O$, we have $e \in g^{-1}O \in \tau$. It follows from the proof of $(b) \Rightarrow (c)$ that there exists a mapping $f \in \mathcal{F}$ such that $SP_{n\in\omega}A_n(f) \subset g^{-1}O$, i.e., $gSP_{n\in\omega}A_n(f) \subset O$. Hence, $O \in \sigma$ and so $\tau \subset \sigma$. Obviously, since $A_k(f) \subset SP_{n\in\omega}A_n(f)$ for every $k \in \omega$ and $f \in \mathcal{F}$, every sequence of $\mathcal{S}$ converges to e in (G, σ). Finally, according to Definition 2.3, $\sigma = \tau_{\mathcal{S}}$, which shows that the implication $(c) \Rightarrow (a)$ holds. □

Lemma 2.5. *Let $\mathcal{S} = \{S_i : i \in I\}$ be a PT-set in a group G (hence, the topology $\tau_{\mathcal{S}}$ exists on G by Theorem 2.4), where $S_i = \{x_n^i\}_{n\in\omega}$ for every $i \in I$, and let p be a homomorphism from $(G, \tau_{\mathcal{S}})$ to a paratopological group H. Then p is continuous if and only if the sequence $p(S_i) = \{p(x_n^i)\}_{n\in\omega}$ converges to the identity e_H in H for every $i \in I$.*

Proof. Necessity is obvious.

Sufficiency. To prove that the homomorphism $p : (G, \tau_{\mathcal{S}}) \to H$ is continuous, it suffices to prove the continuity of p at the identity e_G in $(G, \tau_{\mathcal{S}})$ according to [3, Proposition 1.3.4]. Let $e_H \in U$ with U open in H. By Lemma 2.1, there exists a sequence $\{V_n\}_{n\in\omega}$ consisting of neighborhoods of e_H in H such that $SP_{n\in\omega}V_n \subset U$. By Theorem 2.4, $\{SP_{n\in\omega}A_n(f) : f \in \mathcal{F}\}$ is a base at the identity e_G in $(G, \tau_{\mathcal{S}})$.

Now, let $k \in \omega$ be arbitrary. Since $\{p(x_n^i)\}_{n\in\omega}$ converges to the identity e_H and

$$p(g^{-1}x_n^i g) = p(g)^{-1}p(x_n^i)p(g)$$

for every $g \in G$ and $i \in I$, $\{p(g^{-1}x_n^i g)\}_{n\in\omega}$ also converges to the identity e_H by the joint continuity of multiplication in paratopological groups. So we may construct an $f \in \mathcal{F}$ such that for every $g \in G$ and $i \in I$, $p(g^{-1}x_n^i g) \in V_k$ when $n \geq f(k, i, g)$, whence

$$p(A_k(f)) = p(\bigcup_{i\in I}\bigcup_{g\in G} g^{-1}A^i_{f(k,i,g)}g) \subset V_k.$$

Therefore we have

$$\begin{aligned} p(SP_{n\in\omega}A_n(f)) &= p(\bigcup_{n\in\omega}\bigcup_{\sigma\in\mathbb{S}_{n+1}} A_{\sigma(0)}(f)\cdots A_{\sigma(n)}(f)) \\ &= \bigcup_{n\in\omega}\bigcup_{\sigma\in\mathbb{S}_{n+1}} p(A_{\sigma(0)}(f))\cdots p(A_{\sigma(n)}(f)) \\ &\subset \bigcup_{n\in\omega}\bigcup_{\sigma\in\mathbb{S}_{n+1}} V_{\sigma(0)}\cdots V_{\sigma(n)} \\ &= SP_{n\in\omega}V_n \subset U. \end{aligned}$$

Hence, p is continuous at the identity e_G in $(G, \tau_{\mathcal{S}})$. □

Lemma 2.6. *Let (G, τ) be a T_1 paratopological group and $\mathcal{S}$ be a set of sequences in G. Then the following are equivalent.*

(1) *$\tau = \tau_{\mathcal{S}}$;*
(2) *For every homomorphism p from (G, τ) to a paratopological group H, p is continuous if and only if $p(S)$ converges to the identity e_H in H for every $S \in \mathcal{S}$.*

Proof. (1) ⇒ (2). This holds by Lemma 2.5.

(2) ⇒ (1). Since the identity isomorphism $id_G : (G, \tau) \to (G, \tau)$ is continuous, $id_G(S)$ (namely, S) converges to the identity e_G in G for every $S \in \mathcal{S}$. Thus $\mathcal{S}$ is a PT-set in G. It follows from the definition of $\tau_{\mathcal{S}}$ that $\tau \subset \tau_{\mathcal{S}}$. On the other hand, by hypothesis, the identity isomorphism $id_G : (G, \tau) \to (G, \tau_{\mathcal{S}})$ is also continuous, which shows $\tau_{\mathcal{S}} \subset \tau$. Hence $\tau = \tau_{\mathcal{S}}$. □

Theorem 2.7. *(G, τ) is a T_1 s-paratopological group, i.e., every sequentially continuous homomorphism p from (G, τ) to a paratopological group H is continuous if and only if there exists a PT-set $\mathcal{S}$ in G such that $\tau = \tau_{\mathcal{S}}$.*

Proof. Necessity. Let

$$\mathcal{S} = \{S : S \text{ is a sequence in } G \text{ converging to the identity } e_G \text{ in } (G, \tau)\}.$$

Then $\tau = \tau_{\mathcal{S}}$ according to Lemma 2.6.

Sufficiency directly follows from Lemma 2.6. □

Let us recall the definition of a quotient group. Let G be a paratopological group and H a closed invariant subgroup of G. Denote by G/H the set of all cosets of H in G. Endow G/H the quotient topology τ with respect to the canonical mapping $\pi : G \to G/H$ defined by $\pi(a) = aH$ for every $a \in G$, i.e.,

$$\tau = \{O \subset G/H : \pi^{-1}(O) \text{ is open in } G\}.$$

A natural multiplication in G/H is defined by the rule $xH \cdot yH = xyH$ for all $x, y \in G$. It is well known that this operation $\cdot$ turns G/H with the quotient topology into a paratopological group called the *quotient group* of G with respect to H, and $\pi : G \to G/H$ is a continuous surjective open homomorphism [3].

Theorem 2.8. *Let $\mathcal{S}$ be a PT-set in a group G, H be a closed invariant subgroup of $(G, \tau_{\mathcal{S}})$ and π be the canonical mapping from G onto the quotient group $(G, \tau_{\mathcal{S}})/H$. Then $\pi(\mathcal{S})$ is a PT-set in the abstract group G/H and the identity mapping*

$$id_{G/H} : (G, \tau_{\mathcal{S}})/H \to (G/H, \tau_{\pi(\mathcal{S})})$$

is a topological isomorphism.

Proof. The canonical mapping π being a continuous surjective open homomorphism from $(G, \tau_{\mathcal{S}})$ onto the quotient group $(G, \tau_{\mathcal{S}})/H$, the set $\pi(\mathcal{S})$ is a PT-set in the abstract group G/H.

Denote by τ the topology on the quotient group $(G, \tau_{\mathcal{S}})/H$. By the definition of $\tau_{\pi(\mathcal{S})}$, obviously, $\tau \subset \tau_{\pi(\mathcal{S})}$. In order to prove the theorem, it suffices to show that $\tau_{\pi(\mathcal{S})} \subset \tau$. Assume that $W \in \tau_{\pi(\mathcal{S})}$. If $W \notin \tau$, then $\pi^{-1}(W) \notin \tau_{\mathcal{S}}$. Let

$$\mathcal{B} = \{U \cap \pi^{-1}(V) : U \in \tau_{\mathcal{S}}, V \in \tau_{\pi(\mathcal{S})}\}.$$

Hence, the topology σ generated by the base $\mathcal{B}$ on the abstract group G is strictly finer than the topology $\tau_{\mathcal{S}}$, that is, $\sigma \supset \tau_{\mathcal{S}}$ and $\sigma \neq \tau_{\mathcal{S}}$. It is easy to see that every sequence of $\mathcal{S}$ converges to the identity e in (G, σ). This contradicts the definition of the topology $\tau_{\mathcal{S}}$ on G. Therefore, $W \in \tau$ and $\tau_{\pi(\mathcal{S})} \subset \tau$. □

3. The structure theorem of s-paratopological groups

In this section, we characterize s-paratopological groups making use of free paratopological groups and establish a structure theorem for s-paratopological groups.

A topological space X is called a *Fréchet space* or *Fréchet–Urysohn space* [7] if for every $A \subset X$ and every $x \in \overline{A}$, there exists a sequence $\{x_n\}_{n\in\omega}$ of points of A converging to x. Obviously, every Fréchet space is a sequential space. It was shown [7] that a topological space is sequential if and only if it is a quotient image of a metrizable space.

Definition 3.1. [1] Let κ be an infinite cardinal number. Put

$$X = \{x\} \cup \{x_{\alpha,n} : \alpha < \kappa, n \in \omega\},$$

where the elements of X are mutually distinct. ω^κ denotes the set of all functions from κ to ω. For every $\alpha < \kappa$ and every $l, m \in \omega$, put $W(\alpha, m) = \{x_{\alpha,l} : l \geq m\}$. For every $\alpha < \kappa$ and every $n \in \omega$, let $\mathcal{B}(x_{\alpha,n}) = \{\{x_{\alpha,n}\}\}$. Let

$$\mathcal{B}(x) = \{\{x\} \cup \bigcup_{\alpha<\kappa} W(\alpha, f(\alpha)) : f \in \omega^\kappa\}.$$

The topological space X, generated by the neighborhood system $\{\mathcal{B}(z)\}_{z\in X}$, is called *the fan space* and denoted by S_κ.

It is not difficult to check that the space S_κ is a Fréchet space.

Now we can prove our main theorem.

Theorem 3.2. *Let G be a non-discrete T_1 paratopological group. Then the following statements are equivalent.*

(1) *G is an s-paratopological group.*
(2) *G is topologically isomorphic to a quotient group of a free paratopological group on a metrizable space.*
(3) *G is topologically isomorphic to a quotient group of a free paratopological group on a T_1 Fréchet space.*
(4) *G is topologically isomorphic to a quotient group of a free paratopological group on a T_1 sequential space.*

Proof. (1) $\Rightarrow$ (3). Let (G, τ) be a non-discrete T_1 s-paratopological group. Here, a sequence $\{x_n\}_{n\in\omega}$ converging to the identity e_G in (G, τ) is said to be *non-trivial*, if $x_n \neq x_m$ for any two distinct $n, m \in \omega$ and $x_n \neq e_G$ for every $n \in \omega$. Put

$$\mathcal{S} = \{S : S \text{ is a non-trivial sequence in } (G, \tau)\}.$$

Since (G, τ) is a non-discrete T_1 s-paratopological group, it is easy to see that $\mathcal{S} \neq \emptyset$. Clearly, $\tau \subset \tau_{\mathcal{S}}$ by the definition of $\tau_{\mathcal{S}}$. On the other hand, (G, τ) being a T_1 s-paratopological group, the identity isomorphism $id_G : (G, \tau) \to (G, \tau_{\mathcal{S}})$ is continuous, whence $\tau_{\mathcal{S}} \subset \tau$. So we have $\tau = \tau_{\mathcal{S}}$.

Denote $\mathcal{S} = \{S_i : i \in I\}$, where $|I| = \kappa$ and $S_i = \{x^i_n\}_{n\in\omega}$ for every $i \in I$. For every $i \in I$ and $n \in \omega$, let

$$y^i_n = (x^i_n, i), S'_i = \{y^i_n\}_{n\in\omega} \text{ and } \mathcal{S}' = \{S'_i : i \in I\}.$$

Let

$$X = \{\infty\} \cup \{y^i_n : i \in I, n \in \omega\}$$

be a copy of S_κ. Define a mapping

$$p : X \to (G, \tau)$$

such that

$$p(y^i_n) = x^i_n \text{ for every } i \in I, n \in \omega, \text{ and } p(\infty) = e_G.$$

Then the mapping p is continuous. Indeed, it suffices to prove that p is continuous at the only non-isolated point $\infty \in X$. Because of $\tau = \tau_{\mathcal{S}}$, according to Theorem 2.4, $\{SP_{n\in\omega}A_n(f) : f \in \mathcal{F}\}$ is a base at the identity e_G in (G, τ). Let $e_G \in O$ with O open in (G, τ). Then $SP_{n\in\omega}A_n(f) \subset O$ for some $f \in \mathcal{F}$. Put

$$\varphi : I \to \omega$$

such that for every $i \in I$, $\varphi(i) = f(0, i, e_G)$. Let

$$V_\infty = \{\infty\} \cup \bigcup_{i \in I} \{y_l^i : l \geq \varphi(i)\}.$$

Then V_∞ is an open neighborhood of ∞ in X and

$$p(V_\infty) \subset \bigcup_{i \in I} A^i_{f(0,i,e_G)} \subset A_0(f) \subset SP_{n\in\omega} A_n(f) \subset O.$$

Hence, p is continuous at the point $\infty \in X$.

Since $p : X \to (G, \tau)$ is continuous, we can extend p to a continuous homomorphism $\hat{p} : FP(X) \to (G, \tau)$ by Definition 1.1. Since $p(X) = G$, $\hat{p}$ is an epimorphism. Now we shall prove that $\hat{p}$ is an open homomorphism. It suffices to prove that for every open neighborhood U of the identity $e_{FP(X)}$ in $FP(X)$, $\hat{p}(U)$ contains a neighborhood of e_G in (G, τ). Indeed, by Lemma 2.1, there exists a sequence $\{V_n\}_{n\in\omega}$ consisting of neighborhoods of $e_{FP(X)}$ in $FP(X)$ such that $SP_{n\in\omega} V_n \subset U$. Since $\infty V_n \cap X$ is a neighborhood of ∞ in X, there exists a sequence of functions $\{f_n\}_{n\in\omega}$ from I to ω such that for every $i \in I$, $n \in \omega$,

$$f_n(i) < f_{n+1}(i) \text{ and } \{\infty\} \cup \bigcup_{i\in I} \{y_l^i : l \geq f_n(i)\} \subset \infty V_n.$$

Obviously, for every $i \in I$ and $g \in G$, there exists $j(i, g) \in I$ such that $g^{-1} S_i g = S_{j(i,g)}$. Define a mapping

$$\psi : \omega \times I \times G \to \omega$$

such that for every $n \in \omega, i \in I$ and $g \in G$,

$$\psi(n, i, g) = f_n(j(i, g)).$$

Hence, for every $n \in \omega$,

$$\begin{aligned}
A_n(\psi) &= \bigcup_{i\in I} \bigcup_{g\in G} g^{-1} A^i_{\psi(n,i,g)} g \\
&= \bigcup_{i\in I} \bigcup_{g\in G} \{e_G\} \cup \{g^{-1} x_l^i g : l \geq \psi(n, i, g)\} \\
&= \bigcup_{i\in I} \bigcup_{g\in G} \{e_G\} \cup \{g^{-1} x_l^i g : l \geq f_n(j(i, g))\} \\
&\subset p(\{\infty\} \cup \bigcup_{i\in I} \{y_l^i : l \geq f_n(i)\}) \\
&\subset \hat{p}(\infty V_n) = \hat{p}(V_n).
\end{aligned}$$

Thus $SP_{n\in\omega} A_n(\psi) \subset SP_{n\in\omega} \hat{p}(V_n) \subset \hat{p}(U)$, which shows $\hat{p} : FP(X) \to (G, \tau)$ is a continuous open epimorphism. By the first isomorphism theorem for paratopological groups in [14] (see p. 42), (G, τ) is topologically isomorphic to a quotient group of $FP(X)$.

(3) ⇒ (4). This is clear.

(4) ⇒ (1). At first, we prove that the free paratopological group $FP(X)$ on a T_1 sequential space X is an s-paratopological group. Denote by σ and τ the topologies of $FP(X)$ and X, respectively. Put

$$\mathcal{S} = \{S : S \text{ is a sequence converging to the identity } e \text{ in } (FP(X), \sigma)\}.$$

Obviously, $\mathcal{S}$ is a PT-set in $F_a(X)$ and $\sigma \subset \tau_{\mathcal{S}}$ by the definition of $\tau_{\mathcal{S}}$. Further, $\tau = \sigma|_X \subset \tau_{\mathcal{S}}|_X$. On the other hand, we have $\tau_{\mathcal{S}}|_X \subset \tau$. Indeed, suppose $U \in \tau_{\mathcal{S}}|_X$. If $U \notin \tau$, then $X \setminus U$ is not closed in (X, τ). Since (X, τ) is a sequential space, there exists a sequence $\{x_n\}_{n\in\omega}$ of points of $X \setminus U$ converging to $x \in U$ in (X, τ). Let $S = \{x_n x^{-1}\}_{n\in\omega}$. Then $S \in \mathcal{S}$ and so S converges to the identity e in $(F_a(X), \tau_{\mathcal{S}})$, whence $\{x_n\}_{n\in\omega}$ converges to x in $(F_a(X), \tau_{\mathcal{S}})$. So $x \in X \setminus U$ by $U \in \tau_{\mathcal{S}}|_X$. This is a contradiction. Hence, $\tau_{\mathcal{S}}|_X = \tau$. By Remark 1.2, we have $\sigma \supset \tau_{\mathcal{S}}$ and so $\sigma = \tau_{\mathcal{S}}$. By Theorem 2.7, $FP(X)$ is an s-paratopological group.

Finally, by Theorem 2.8, G is also an s-paratopological group.

(2) ⇒ (4). This is obvious.

(4) ⇒ (2). Again, by the first isomorphism theorem for paratopological groups in [14] (see p. 42), it suffices to prove that the free paratopological group on a sequential space is the image of the free paratopological group on a metrizable space under a continuous open homomorphism. Indeed, let Y be a sequential space. Then there exists a quotient onto mapping $q : M \to Y$, where M is a metrizable space. Then f admits an extension to the continuous open homomorphism $\hat{f} : FP(M) \to FP(Y)$ by [4, Lemma 4.7] or [13, Proposition 2.10], which completes the proof of (4) ⇒ (2). □

Free Abelian paratopological groups were introduced in [17] analogously to free paratopological groups.

Definition 3.3. [17] Let X be a subspace of an Abelian paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X \rangle = G$; and
(2) every continuous mapping $f : X \to H$ of X to an arbitrary Abelian paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free Abelian paratopological group* (briefly, *free Abelian paratopological group*) on X and is denoted by $AP(X)$.

In a way similar to Theorem 3.2, we have the following Abelian case of it.

Theorem 3.4. *Let G be a non-discrete T_1 Abelian paratopological group. Then the following statements are equivalent.*

(1) *G is an s-paratopological group.*
(2) *G is topologically isomorphic to a quotient group of a free Abelian paratopological group on a metrizable space.*
(3) *G is topologically isomorphic to a quotient group of a free Abelian paratopological group on a T_1 Fréchet space.*
(4) *G is topologically isomorphic to a quotient group of a free Abelian paratopological group on a T_1 sequential space.*

We conclude this paper with a natural question.

Question 3.5. Does there exist an s-topological group which is not an s-paratopological group?

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