Topology and its Applications 230 (2017) 267–275

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings of the First Pan-Pacific International Conference on Topology and Applications (PPICTA)

On pairwise k-semi-stratifiable spaces [☆]

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A R T I C L E I N F O

Article history:
Received 5 January 2016
Received in revised form 14 October 2016
Accepted 23 October 2016
Available online 24 August 2017

MSC:
primary 54E55
secondary 54E20, 54C08, 26A15

Keywords:
Bitopological spaces
Quasi-metrizable spaces
Pairwise g-functions
Pairwise stratifiable spaces
Pairwise k-semi-stratifiable spaces
Semi-continuous functions

A B S T R A C T

In this paper, the concept of a pairwise k-semi-stratifiable space is introduced and studied. Some characterizations of pairwise k-semi-stratifiable spaces by means of pairwise g-functions and semi-continuous functions are given. A new characterization of quasi-pseudo-metrizable spaces is obtained by using pairwise k-semi-stratifiable spaces, which improves a theorem of Marín in [17].

1. Introduction

The study of bitopological spaces was first initiated by Kelly [8]. Since then many works on bitopological spaces have been done by several authors: Datta [1], Fletcher et al. [3], Ganster and Reilly [4], Kovár [9], Künzi and Mushaandja [10], Lane [11], Marín [17], Raghavan and Reilly [20], Srivastava and Bhatia [21] and others. A large number of papers have been done, in order to generalize the topological concepts to the bitopological setting. In [10], Künzi and Mushaandja obtained characterizations of some topological ordered spaces and generalized metric spaces. It is well known that stratifiable spaces form one of interesting classes of generalized metric spaces. This notion has been generalized to bitopological spaces [6], and many properties have been extended. Marín and Romaguera [18] introduced the notion of monotonically normal bitopological

☆ The project is supported by the NNSF (Nos. 11471153, 61379021, 11571158) and the NSF of Fujian Province of China (No. 2013J01029).
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http://dx.doi.org/10.1016/j.topol.2017.08.039
0166-8641/

spaces (or pairwise monotonically normal spaces) which is a useful generalization of pairwise stratifiable spaces, and characterized pairwise monotonically normal spaces in terms of a mixed condition of insertions and extensions of semi-continuous functions. It is well known that a topological space is stratifiable if and only if it is monotonically normal and semi-stratifiable [7]. A bitopological space is pairwise stratifiable if and only if it is pairwise monotonically normal and pairwise semi-stratifiable [18]. In [12] (resp. [14]), Li (resp. Li and Lin) gave some characterizations of pairwise stratifiable spaces (resp. pairwise semi-stratifiable spaces) by means of pairwise g-functions and semi-continuous functions. In this paper, we introduce the notion of pairwise k-semi-stratifiable bitopological spaces, which is a generalization of pairwise stratifiable spaces, and every pairwise k-semi-stratifiable space is pairwise semi-stratifiable. Our main purpose is to address the following question.

Question 1.1. *How to characterize pairwise k-semi-stratifiable spaces in terms of a mixed condition of insertions and extensions of semi-continuous functions?*

In Section 2, we give a result on the characterizations of pairwise k-semi-stratifiable spaces similar to that of stratifiable spaces in [12] and pairwise semi-stratifiable spaces in [14]. This result provides an answer to Question 1.1. We then give some applications of this result. In Section 3, we show that a bitopological space (X, τ_1, τ_2) is quasi-pseudo-metrizable if and only if it is pairwise k-semi-stratifiable and each (X, τ_i) is a γ-space. This is a generalization of the following main theorem in [17]: A bitopological space (X, τ_1, τ_2) is quasi-pseudo-metrizable if and only if it is pairwise stratifiable and each (X, τ_i) is a γ-space.

Throughout this paper, all topological spaces are T_1.

By a bitopological space (X, τ_1, τ_2), it is meant a nonempty set X equipped with two topologies τ_1 and τ_2. In the rest of the paper, when we are concerned at the topologies τ_i and τ_j, we suppose $i, j = 1, 2$, and $i \neq j$. Put $\tau_i^c = \{X - O : O \in \tau_i\}$, and write $\mathrm{cl}_{\tau_i} A$ for the closure of A in the topological space (X, τ_i). The set of all positive integers is denoted by $\mathbb{N}$. We refer the readers to [2,16] for undefined terms.

A real-valued function f defined on a topological space (X, τ_i) is *τ_i-lower (resp. τ_i-upper) semi-continuous* if for each $x \in X$ and each real number r with $f(x) > r$ (resp. $f(x) < r$), there exists a τ_i-open set $U \subseteq X$ such that $x \in U$ and $f(x') > r$ (resp. $f(x') < r$) for every $x' \in U$. We write $LSC_{\tau_i}(X)$ (resp. $USC_{\tau_i}(X)$) for the set of all real-valued τ_i-lower (resp. τ_i-upper) semi-continuous functions on X into $I = [0, 1]$. Let f be a real-valued τ_i-lower (resp. τ_i-upper) semi-continuous function defined on a topological space (X, τ_i), then $\{x \in X : f(x) \leqslant r\} \in \tau_i^c$ (resp. $\{x \in X : f(x) \geqslant r\} \in \tau_i^c$) for any real number r.

Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. We write χ_A for the *characteristic function* on A, that is, a function $\chi_A : X \to [0, 1]$ is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then $\chi_A \in USC_{\tau_i}(X)$ if and only if $A \in \tau_i^c$, and $\chi_A \in LSC_{\tau_i}(X)$ if and only if $A \in \tau_i$.

Definition 1.2. A bitopological space (X, τ_1, τ_2) is called *pairwise stratifiable* [6] if for each $A \in \tau_i^c$ $(i = 1, 2)$, one can assign a sequence $\{D_{ij}(n, A)\}_{n \in \mathbb{N}}$ of τ_j-open sets ($j = 1, 2$ and $j \neq i$) such that

(1) $A = \bigcap_{n \in \mathbb{N}} D_{ij}(n, A) = \bigcap_{n \in \mathbb{N}} \mathrm{cl}_{\tau_i} D_{ij}(n, A)$;
(2) if $A \subseteq E \in \tau_i^c$, then $D_{ij}(n, A) \subseteq D_{ij}(n, E)$ for each $n \in \mathbb{N}$.

A bitopological space (X, τ_1, τ_2) is called *pairwise semi-stratifiable* [19] if D_{ij} $(i, j = 1, 2$ and $i \neq j)$ satisfies (2) and

(1') $A = \bigcap_{n \in \mathbb{N}} D_{ij}(n, A)$ for each $A \in \tau_i^c$.

Definition 1.3. A bitopological space (X, τ_1, τ_2) is called *pairwise k-semi-stratifiable* if for each $A \in \tau_i^c$ $(i = 1, 2)$, one can assign a sequence $\{D_{ij}(n, A)\}_{n\in\mathbb{N}}$ of τ_j-open sets ($j = 1, 2$ and $j \neq i$) such that

(1) $A = \bigcap_{n\in\mathbb{N}} D_{ij}(n, A)$;
(2) if $A \subseteq E \in \tau_i^c$, then $D_{ij}(n, A) \subseteq D_{ij}(n, E)$ for each $n \in \mathbb{N}$;
(3) if K is τ_i-compact in X with $K \cap A = \emptyset$, then $K \cap D_{ij}(m, A) = \emptyset$ for some $m \in \mathbb{N}$.

It follows from Definition 1.3 that a bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if there exists an operator $D_{ij} : \mathbb{N} \times \tau_i^c \to \tau_j$ ($i, j = 1, 2$ and $i \neq j$) satisfying the conditions (1) $\sim$ (3) in Definition 1.3.

We may assume that the operator D_{ij} is monotonic with respect to $n \in \mathbb{N}$, that is, $D_{ij}(n + 1, A) \subseteq D_{ij}(n, A)$ for each $n \in \mathbb{N}$ and each $A \in \tau_i^c$.

The following lemma, included for convenience, is clearly just another way of stating the definition.

Lemma 1.4. *A bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if for each $U \in \tau_i$ $(i = 1, 2)$, one can assign a sequence $\{F_{ij}(n, U)\}_{n\in\mathbb{N}}$ of τ_j-closed sets ($j = 1, 2$ and $j \neq i$) such that*

(1) *$U = \bigcup_{n\in\mathbb{N}} F_{ij}(n, U)$;*
(2) *if $U \subseteq V \in \tau_i$, then $F_{ij}(n, U) \subseteq F_{ij}(n, V)$ for each $n \in \mathbb{N}$;*
(3) *if K is τ_i-compact in X with $K \subseteq U$, then $K \subseteq F_{ij}(m, U)$ for some $m \in \mathbb{N}$.*

We may assume that the operator $F_{ij} : \mathbb{N} \times \tau_i \to \tau_j^c$ is monotonic with respect to $n \in \mathbb{N}$, that is, $F_{ij}(n, U) \subseteq F_{ij}(n + 1, U)$ for each $n \in \mathbb{N}$ and each $V \in \tau_i$.

2. Characterizations of pairwise *k*-semi-stratifiable spaces

First, we give some characterizations of pairwise k-semi-stratifiable spaces by using pairwise g-functions. A *pairwise g-function* on a bitopological space (X, τ_1, τ_2) is a pair of functions (g_1, g_2) such that for $i = 1, 2$, $g_i : \mathbb{N} \times X \to \tau_i$ satisfies $x \in g_i(n, x)$ and $g_i(n + 1, x) \subseteq g_i(n, x)$ for each $x \in X$ and $n \in \mathbb{N}$. In this paper we put $g_i(n, A) = \bigcup\{g_i(n, x) : x \in A\}$ for $i = 1, 2$, and each $A \subseteq X$.

Theorem 2.1. *For a bitopological space (X, τ_1, τ_2), consider the following conditions:*

(1) *X is pairwise k-semi-stratifiable;*
(2) *there exists a pairwise g-function (g_1, g_2) which satisfies that for $i, j = 1, 2$ and $i \neq j$:*
   (i) *if $F \in \tau_i^c$, then $F = \bigcap_{n\in\mathbb{N}} g_j(n, F)$;*
   (ii) *if K is τ_i-compact in X and $F \in \tau_i^c$ with $K \cap F = \emptyset$, then $K \cap g_j(m, F) = \emptyset$ for some $m \in \mathbb{N}$;*
(3) *there exists a pairwise g-function (g_1, g_2) which satisfies that for $i, j = 1, 2$ and $i \neq j$, each $x \in X$ and two sequences $\{x_n\}, \{y_n\}$ of X, if $x_n \in g_j(n, y_n)$ and $\{x_n\}$ τ_i-converges to x, then $\{y_n\}$ τ_i-converges to x.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3), and (1) $\Leftrightarrow$ (3) if every (X, τ_i) is a T_2-space.

Proof. (1) $\Rightarrow$ (2). Let (X, τ_1, τ_2) be a pairwise k-semi-stratifiable space and D_{ij} ($i, j = 1, 2$ and $i \neq j$) operators on X which satisfy the conditions (1) $\sim$ (3) in Definition 1.3. For each $x \in X$, let $g_j(n, x) = \bigcap_{k\leqslant n} D_{ij}(k, \{x\})$. Then (g_1, g_2) is a pairwise g-function on (X, τ_1, τ_2) satisfying the required conditions.

(2) $\Rightarrow$ (1). Suppose (2) holds. For each $F \in \tau_i^c$, each $n \in \mathbb{N}$, $i, j = 1, 2$ and $i \neq j$, define $D_{ij}(n, F) = g_j(n, F)$, then $D_{ij}(n, F) \in \tau_j$. The function $D_{ij} : \mathbb{N} \times \tau_i^c \to \tau_j$ defined in this way is a pairwise

k-semi-stratifiable operator in (X, τ_1, τ_2). In fact, the conditions (1) and (2) in Definition 1.3 clearly hold. We prove that the condition (3) in Definition 1.3 is also true. If K is τ_i-compact in X and $F \in \tau_i^c$ with $K \cap F = \emptyset$, then $K \cap g_j(m, F) = \emptyset$ for some $m \in \mathbb{N}$. Thus $K \cap D_{ij}(m, F) = \emptyset$ for some $m \in \mathbb{N}$.

(1) $\Rightarrow$ (3). Let (X, τ_1, τ_2) be a pairwise k-semi-stratifiable space and F_{ij} $(i, j = 1, 2$ and $i \neq j)$ operators on X which satisfy the conditions (1) $\sim$ (3) in Lemma 1.4. For each $n \in \mathbb{N}$ and each $x \in X$, let $g_j(n, x) = X - F_{ij}(n, X - \{x\})$. Then (g_1, g_2) is a pairwise g-function on X satisfying the required conditions. In fact, let $x_n \in g_j(n, y_n)$ for each $n \in \mathbb{N}$ and $\{x_n\}$ τ_i-converge to $x \in U \in \tau_i$. We can assume $x_n \in U$ for each $n \in \mathbb{N}$. There exists $m \in \mathbb{N}$ such that the τ_i-compact subset $\{x\} \cup \{x_n : n \in \mathbb{N}\} \subseteq F_{ij}(m, U)$ by the condition (3) in Lemma 1.4. Therefore, for all $n \geqslant m$, $x_n \in F_{ij}(n, U) - F_{ij}(n, X - \{y_n\})$, thus $U \nsubseteq X - \{y_n\}$ by the condition (3) in Lemma 1.4, i.e., $y_n \in U$. This shows that $\{y_n\}$ τ_i-converges to x.

Next, assume that (X, τ_i) is a T_2-space for each $i = 1, 2$. We will prove that (3) $\Rightarrow$ (1). Suppose there exists a pairwise g-function (g_1, g_2) which satisfies the condition (3) in the theorem. Let $F \in \tau_i^c$, and define $D_{ij}(n, F) = g_j(n, F)$ for each $n \in \mathbb{N}$, $i, j = 1, 2$ and $i \neq j$. Then $D_{ij}(n, F) \in \tau_j$. We will verify that D_{ij} $(i, j = 1, 2$ and $i \neq j)$ are operations on X which satisfy the conditions (1) $\sim$ (3) in Definition 1.3. In fact, it is clear that (2) holds. For (1), we need only prove that $\bigcap_{n\in\mathbb{N}} D_{ij}(n, F) \subseteq F$. Let $x \in \bigcap_{n\in\mathbb{N}} D_{ij}(n, F)$. Then for each $n \in \mathbb{N}$, there is $y_n \in F$ such that $x \in g_j(n, y_n)$. Since the constant sequence $\{x_n\}$ with $x_n = x$ τ_i-converges to x, by Theorem 2.1(3), $\{y_n\}$ also τ_i-converges to x. As $F \in \tau_i^c$, we conclude that $x \in F$. For (3), let K be τ_i-compact in X, $F \in \tau_i^c$ and $K \cap F = \emptyset$. If $K \cap g_j(n, F) \neq \emptyset$ for each $n \in \mathbb{N}$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in K \cap g_j(n, y_n)$ and $y_n \in F$ for each $n \in \mathbb{N}$. Since (X, τ_i) is a T_2-space and K is a τ_i-compact subset of X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ τ_i-converges to $x \in K$, thus $\{y_{n_k}\}$ τ_i-converges to $x \in F$, which is a contradiction. □

Next, we characterize the pairwise k-semi-stratifiability of bitopological spaces by means of extensions of semi-continuous functions.

Theorem 2.2. *A bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if for each partially ordered set $(\mathbb{H}, \leqslant)$ and a map $H_i : \mathbb{N} \times \mathbb{H} \to \tau_i^c$ $(i = 1, 2)$ satisfying the following (1) and (2), there is a map $G_j : \mathbb{N} \times \mathbb{H} \to \tau_j$ with (1) and (2) for G_j such that $H_i(n, h) \subseteq G_j(n, h)$ for all $h \in \mathbb{H}$, $n \in \mathbb{N}$ $(j = 1, 2$ and $j \neq i)$, $\bigcap_{n\in\mathbb{N}} H_i(n, h) = \bigcap_{n\in\mathbb{N}} G_j(n, h)$ for all $h \in \mathbb{H}$, and the following (3) holds.*

(1) *$H_i(n + 1, h) \subseteq H_i(n, h)$ for each $h \in \mathbb{H}$ and each $n \in \mathbb{N}$;*
(2) *for every $h_1, h_2 \in H_i$, if $h_1 \leqslant h_2$, then $H_i(n, h_2) \subseteq H_i(n, h_1)$ for each $n \in \mathbb{N}$;*
(3) *if K is τ_i-compact in X, and $K \cap H_i(n, h) = \emptyset$ for some $n \in \mathbb{N}$ and $h \in \mathbb{H}$, then there exists $m \in \mathbb{N}$ such that $K \cap G_j(m, h) = \emptyset$.*

Proof. Necessity. Let (X, τ_1, τ_2) be a pairwise k-semi-stratifiable space and an operator F_{ij} be defined by Lemma 1.4 for every $i, j = 1, 2$ and $i \neq j$. We show that the map $G_j : \mathbb{N} \times \mathbb{H} \to \tau_j$ defined by $G_j(n, h) = X - F_{ij}(n, X - H_i(n, h))$ satisfies the conditions of the theorem. By the properties of F_{ij} and H_i, one can easily verify that the conditions (1) and (2) hold for G_j. By the condition (1) in Lemma 1.4, the equality $\bigcup_{n\in\mathbb{N}} F_{ij}(n, V) = V$ holds for all $V \in \tau_i$, and therefore, we have $F_{ij}(n, V) \subseteq V$ for each $n \in \mathbb{N}$. Since $H_i(n, h) \in \tau_i^c$, then $X - H_i(n, h) \in \tau_i$ and $X - H_i(n, h) \supseteq F_{ij}(n, X - H_i(n, h))$, so $H_i(n, h) \subseteq G_j(n, h)$ for each $h \in \mathbb{H}$ and each $n \in \mathbb{N}$.

We show that $\bigcap_{n\in\mathbb{N}} H_i(n, h) \supseteq \bigcap_{n\in\mathbb{N}} G_j(n, h)$ for each $h \in \mathbb{H}$. If $x \notin \bigcap_{n\in\mathbb{N}} H_i(n, h)$, then $x \notin H_i(m_0, h)$ for some $m_0 \in \mathbb{N}$. Consequently, $x \in F_{ij}(n_0, X - H_i(m_0, h))$ for some $n_0 \in \mathbb{N}$ by $X - H_i(m_0, h) = \bigcup_{n\in\mathbb{N}} F_{ij}(n, X - H_i(m_0, h))$. Let $m = \max\{n_0, m_0\}$. Then

$$x \in F_{ij}(n_0, X - H_i(m_0, h)) \subseteq F_{ij}(m, X - H_i(m_0, h)) \subseteq F_{ij}(m, X - H_i(m, h)).$$

Since $F_{ij}(m, X - H_i(m,h)) \cap G_j(m,h) = \emptyset$, then $x \notin G_j(m,h)$, so $x \notin \bigcap_{n\in\mathbb{N}} G_j(n,h)$, which proves $\bigcap_{n\in\mathbb{N}} H_i(n,h) = \bigcap_{n\in\mathbb{N}} G_j(n,h)$ holds for all $h \in \mathbb{H}$.

We shall show that (3) holds. Let K be τ_i compact in X with $K \cap H_i(n,h) = \emptyset$ for some $n \in \mathbb{N}$ and $h \in \mathbb{H}$. Since $K \subseteq X - H_i(n,h)$, by Lemma 1.4, $K \subseteq F_{ij}(k_1, X - H_i(n,h))$ for some $k_1 \in \mathbb{N}$. Let $k = \max\{n, k_1\}$. Then

$$K \subseteq F_{ij}(k_1, X - H_i(n,h)) \subseteq F_{ij}(k, X - H_i(k,h)) = X - G_j(k,h).$$

Hence $K \cap G_j(k,h) = \emptyset$.

Sufficiency. Let $\mathbb{H} = \tau_i$, $i = 1, 2$, and define a partial order $\leqslant$ on $\mathbb{H}$ by $h_1 \leqslant h_2 \Leftrightarrow h_1 \subseteq h_2$ for each pair $h_1, h_2 \in \mathbb{H}$. We consider the map $H_i : \mathbb{N} \times \tau_i \to \tau_i^c$ defined by $H_i(n,U) = X - U$ for each $U \in \tau_i$ and $n \in \mathbb{N}$. One can easily verify that H_i satisfies the conditions (1) and (2) in the theorem. So there is a map $G_j : \mathbb{N} \times \tau_i \to \tau_j$, $j = 1, 2$ and $j \neq i$, such that the conditions (1) $\sim$ (3) hold for G_j. Moreover, $H_i(n,V) \subseteq G_j(n,V)$ for all $n \in \mathbb{N}$ and all $V \in \tau_i$, and $\bigcap_{n\in\mathbb{N}} H_i(n,V) = \bigcap_{n\in\mathbb{N}} G_j(n,V)$. Let $F_{ij}(n,V) = X - G_j(n,V)$. Then the sequence $\{F_{ij}(n,V)\}_{n\in\mathbb{N}}$ satisfies the conditions in Lemma 1.4. In fact, it is clear that the condition (2) holds in Lemma 1.4. Since $H_i(n,V) \subseteq G_j(n,V)$, we have

$$F_{ij}(n,V) = X - G_j(n,V) \subseteq X - H_i(n,V) = V$$

for each $n \in \mathbb{N}$ and

$$\begin{aligned} V &= X - \bigcap_{n\in\mathbb{N}} H_i(n,V) = X - \bigcap_{n\in\mathbb{N}} G_j(n,V) \\ &= \bigcup_{n\in\mathbb{N}} (X - G_j(n,V)) = \bigcup_{n\in\mathbb{N}} F_{ij}(n,V). \end{aligned}$$

Therefore the condition (1) holds in Lemma 1.4.

For a τ_i-compact subset $K \subseteq U \in \tau_i$, since $K \cap H_i(n,U) = K - U = \emptyset$, then $K \cap G_j(m,U) = \emptyset$ for some $m \in \mathbb{N}$ by the condition (3) of our assumption, i.e., $K \subseteq F(m,U)$. Therefore the condition (3) holds in Lemma 1.4. Hence (X, τ_1, τ_2) is a pairwise k-semi-stratifiable space. □

Let $\mathbb{H}$ be the set $LSC_{\tau_i}(X)$ or $USC_{\tau_i}(X)$ for $i = 1, 2$. Define a partial order $\leqslant$ on $\mathbb{H}$ for each pair $h_1, h_2 \in \mathbb{H}$ by $h_1 \leqslant h_2 \Leftrightarrow h_1(x) \leqslant h_2(x)$ for each $x \in X$.

Theorem 2.3. *A bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if for each function $f \in LSC_{\tau_i}(X)$ $(i = 1, 2)$, one assigns a function $h(f) \in USC_{\tau_j}(X)$ $(j = 1, 2$ and $j \neq i)$ such that*

(1) *$0 \leqslant h(f) \leqslant f$ and $0 < h(f)(x) < f(x)$ whenever $f(x) > 0$;*
(2) *$h(f) \leqslant h(f')$ whenever $f \leqslant f'$;*
(3) *if K is a τ_i-compact subset of X and there exists $f \in LSC_{\tau_i}(X)$ such that $f(x) > 0$ for all $x \in K$, then $h(f)(x) > r$ for some $0 < r < 1$ and all $x \in K$.*

Proof. Suppose that (X, τ_1, τ_2) is pairwise k-semi-stratifiable. For each $n \in \mathbb{N}$ and $f \in LSC_{\tau_i}(X)$ $(i = 1, 2)$, let

$$H_i(n,f) = \{x \in X : f(x) \leqslant 1/2^{n-1}\}.$$

Then $H_i(n,f) \in \tau_i^c(X)$. So a map $H_i : \mathbb{N} \times LSC_{\tau_i}(X) \to \tau_i^c$ is defined by the equality and it is easy to verify that H_i satisfies the conditions (1) and (2) in Theorem 2.2. Since (X, τ_1, τ_2) is a pairwise k-semi-stratifiable space, there exists a map $G_j : \mathbb{N} \times LSC_{\tau_i}(X) \to \tau_j$ ($j = 1, 2$ and $j \neq i$) such that the conditions (1) $\sim$ (3) in Theorem 2.2 hold for G_j. By the conditions in Theorem 2.2, we have

$$\bigcap_{n\in\mathbb{N}} H_i(n,f) = \bigcap_{n\in\mathbb{N}} G_j(n,f) = \{x \in X : f(x) = 0\}. \qquad (*)$$

Now, let $\alpha(n,f) = \chi_{G_j(n,f)}$. Since $G_j(n,f) \in \tau_j$, we have $\alpha(n,f) \in LSC_{\tau_j}(X)$. Let

$$h(f)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,f)(x),\ x \in X.$$

Then $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,f) \in LSC_{\tau_j}(X)$ by [22, Theorem 2.4], so $h(f) \in USC_{\tau_j}(X)$.

We shall show that the map h defined above satisfies the conditions (1) $\sim$ (3) in the theorem. Take any $x \in X$. If $f(x) = 0$, then $x \in H_i(n,f) \subseteq G_j(n,f)$ and so $\alpha(n,f)(x) = 1$ for all $n \in \mathbb{N}$ by $(*)$. Therefore,

$$h(f)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,f)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} = 0.$$

If $f(x) > 0$, then $x \notin \bigcap_{n\in\mathbb{N}} G_j(n,f)$ by $(*)$. Let $k = \min\{n \in \mathbb{N} : x \notin G_j(n,f)\}$. Then for all $n < k$, $x \in G_j(n,f)$ and so $\alpha(n,f)(x) = 1$. But $x \notin G_j(k,f)$, and so $x \notin H_i(k,f)$ by $H_i(k,f) \subset G_j(k,f)$. This implies that $f(x) > 1/2^{k-1}$. Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,f)(x) &= \sum_{n=1}^{k-1} \frac{1}{2^n}\alpha(n,f)(x) + \sum_{n=k}^{\infty} \frac{1}{2^n}\alpha(n,f)(x) \\ &= 1 - \frac{1}{2^{k-1}}.\end{aligned}$$

By the definition of h, we have $0 < h(f)(x) \leqslant \frac{1}{2^{k-1}} < f(x)$. Therefore, (1) holds.

Next, we shall show that $h(f_1) \leqslant h(f_2)$ whenever $f_1 \leqslant f_2$. Suppose that $f_1 \leqslant f_2$, then $G_j(n,f_2) \subseteq G_j(n,f_1)$, and so $\alpha(n,f_2) \leqslant \alpha(n,f_1)$ for each $n \in \mathbb{N}$. It follows from the definition of h that $h(f_1) \leqslant h(f_2)$. Therefore, (2) holds.

It remains to show that the condition (3) holds. Let K be a τ_i-compact subset of X, and suppose that there exists an $f \in LSC_{\tau_i}(X)$ such that $f(x) > 0$ for all $x \in K$. Then $K \cap (\bigcap_{n\in\mathbb{N}} H_i(n,f)) = \emptyset$ by $(*)$. Since K is τ_i-compact, there exists $n_0 \in \mathbb{N}$ such that $K \cap H_i(n_0,f) = \emptyset$. By the condition (3) of Theorem 2.2, there exists $m \in \mathbb{N}$ such that $K \cap G_j(m,f) = \emptyset$. Thus $\alpha(m,f)(x) = 0$ for all $x \in K$, so $\alpha(n,f)(x) = 0$ for all $n \geqslant m$ and all $x \in K$. Take

$$2r = 1 - \sum_{n=1}^{m-1} \frac{1}{2^n}.$$

Then $0 < r < 1$ and for each $x \in K$,

$$\begin{aligned}h(f)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) &= 1 - \sum_{n=1}^{m-1} \frac{1}{2^n}\alpha(n,h)(x) \\ &\geqslant 1 - \sum_{n=1}^{m-1} \frac{1}{2^n} > r.\end{aligned}$$

Conversely, suppose that there is a map $h : LSC_{\tau_i}(X) \to USC_{\tau_j}(X)$ $(i, j = 1, 2$ and $i \neq j)$ that satisfies the conditions (1) $\sim$ (3) in the theorem. For any fixed $V \in \tau_i$, we consider the function $f_V = \chi_V$. Then $f_V \in LSC_{\tau_i}(X)$ and so $h(f_V) \in USC_{\tau_j}(X)$. For each $n \in \mathbb{N}, i, j = 1, 2$ and $i \neq j$, let

$$F_{ij}(n, V) = \{x \in X : h(f_V)(x) \geqslant 1/2^n\}.$$

Then a map $F_{ij} : \mathbb{N} \times \tau_i \to \tau_j^c$ is defined by the equality above. We shall show that the map F_{ij} satisfies the conditions (1) $\sim$ (3) in Lemma 1.4.

For each $n \in \mathbb{N}$ and $x \in X$, if $x \notin V$, then $f_V(x) = 0$. Also $x \notin F_{ij}(n, V)$ for all $n \in \mathbb{N}$ by the condition $0 \leqslant h(f_V) \leqslant f_V$ in the theorem. Hence $F_{ij}(n, V) \subseteq V$ for all $n \in \mathbb{N}$ and so $\bigcup_{n\in\mathbb{N}} F_{ij}(n, V) \subseteq V$. Conversely, for each $x \in V$, we have $f_V(x) = \chi_V(x) = 1 > 0$, and so $h(f_V)(x) > 0$ by the condition (1) given in the theorem. Hence there is $m \in \mathbb{N}$ such that $h(f_V)(x) \geqslant \frac{1}{2^m}$, which implies that $x \in F_{ij}(m, V)$. Therefore, $V \subseteq \bigcup_{n\in\mathbb{N}} F_{ij}(n, V)$. Hence the condition (1) in Lemma 1.4 holds.

If $U, V \in \tau_i$, and $U \subseteq V$, then $f_U \subseteq f_V$. Also $h(f_U) \leqslant h(f_V)$ by the condition (2) given in the theorem. Thus $F_{ij}(n, U) \subseteq F_{ij}(n, V)$ for all $n \in \mathbb{N}$. Therefore, the condition (2) holds in Lemma 1.4.

Suppose that a τ_i-compact subset $K \subseteq U \in \tau_i$. Then $f_U = \chi_U \in LSC_{\tau_i}(X)$ and $f_U(x) = 1$ for all $x \in K$. By (3) of our assumption, there exists $m_0 \in \mathbb{N}$ such that $h(f_U)(x) > 1/2^{m_0}$ for all $x \in K$, then $K \subseteq F_{ij}(m_0, U)$. Hence F_{ij} satisfies all conditions in Lemma 1.4, so (X, τ_1, τ_2) is pairwise k-semi-stratifiable. □

Corollary 2.4. *A bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if for each pair (A, U) of subsets of X with $A \in \tau_j^c$, $U \in \tau_i$ $(i, j = 1, 2$ and $i \neq j)$ and $A \subseteq U$, there is an $h_{A,U} \in LSC_{\tau_j}(X)$ such that*

(1) $A = h_{A,U}^{-1}(0)$, $X - U = h_{A,U}^{-1}(1)$;
(2) $h_{A,U} \geqslant h_{B,V}$ *whenever* $A \subseteq B \in \tau_j^c$ *and* $U \subseteq V \in \tau_i$;
(3) *if K is a τ_i-compact subset with $K \subseteq U$, then there exists some $0 < r < 1$ such that $h_{\emptyset,U}(x) < r$ for all $x \in K$.*

Proof. Necessity. Suppose that (X, τ_1, τ_2) is a pairwise k-semi-stratifiable space. Then, by Theorem 2.3, there is a map $h : LSC_{\tau_i}(X) \to USC_{\tau_j}(X)$ $(i, j = 1, 2$ and $i \neq j)$ that satisfies the conditions (1) $\sim$ (3) in Theorem 2.3. For each pair (A, U) of subsets of X with $A \in \tau_j^c$, $U \in \tau_i$ $(i, j = 1, 2$ and $j \neq i)$ and $A \subset U$, let $f_A = 1 - \chi_A$, $g_U = h(\chi_U)$. Since $A \in \tau_j^c$ and $U \in \tau_i$, we have $\chi_A \in USC_{\tau_j}(X)$ and $\chi_U \in LSC_{\tau_i}(X)$. Therefore, $f_A \in LSC_{\tau_j}(X)$ and $g_U \in USC_{\tau_j}(X)$. Define $h_{A,U} : X \to [0, 1]$ by $h_{A,U} = \frac{f_A}{1+g_U}$. Then $h_{A,U} \in LSC_{\tau_j}(X)$ by Proposition 2.2 in [22]. It is easy to verify that $h_{A,U} \geqslant h_{B,V}$ whenever $A \subseteq B \in \tau_j^c$ and $U \subseteq V \in \tau_i$.

From the definition of $h_{A,U}$, one can see that $h_{A,U}(x) = 0$ if and only if $x \in A$, which implies that $A = h_{A,U}^{-1}(0)$. Similarly, we can verify that $X - U = h_{A,U}^{-1}(1)$.

Let K be a τ_i-compact subset with $K \subseteq U \in \tau_i$. Then $\chi_U \in LSC_{\tau_i}(X)$ and $\chi_U(x) > 0,\ x \in K$. Thus $h(\chi_U)(x) > t$ for some $t > 0$ and all $x \in K$ by (3) in Theorem 2.3. Let $r = \frac{1}{1+t}$, then $0 < r < 1$ and $h_{\emptyset,U}(x) < r$ for all $x \in K$.

Sufficiency. For each $U \in \tau_i$ and $i = 1, 2$, let $f_U = 1 - h_{\emptyset,U}$. Then for every $j = 1, 2$ and $j \neq i$, $f_U \in USC_{\tau_j}(X)$, and $f_U \leqslant h_V$ when $U \subseteq V \in \tau_i$. It is easy to verify that $f_U(x) = 0$ if and only if $x \notin U$. For each $n \in \mathbb{N}$, $j, i = 1, 2$ and $j \neq i$, let $F_{ij}(n, U) = \{x \in X : f_U(x) \geqslant \frac{1}{2^n}\}$, and hence $F_{ij}(n, U) \in \tau_j^c$ by $f_U \in USC_{\tau_j}(X)$. To prove that (X, τ_1, τ_2) is pairwise k-semi-stratifiable, it suffices to show that the F_{ij} satisfies the conditions (1) $\sim$ (3) in Lemma 1.4. This can be done by a method similar to that used in the proof of Theorem 2.3. □

Corollary 2.5. *A bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable if and only if for every $i = 1, 2$ and $U \in \tau_i$ in X there is $f_U \in USC_{\tau_j}(X)$ for every $j = 1, 2$ and $j \neq i$, such that*

(1) *$X - U = f_U^{-1}(0)$;*
(2) *$f_U \leqslant f_V$ whenever $U \subseteq V \in \tau_i$;*
(3) *if K is a τ_i-compact subset with $K \subseteq U$, then there exists some $0 < r < 1$ such that $f_U(x) > r$ for all $x \in K$.*

Proof. Suppose that (X, τ_1, τ_2) is a pairwise k-semi-stratifiable space. For every $i = 1, 2$ and $U \in \tau_i$, let $f_U = 1 - h_{\emptyset,U}$, where $h_{\emptyset,U}$ is the function given in Corollary 2.4. Obviously, for every $j = 1, 2$ and $j \neq i$, $f_U \in USC_{\tau_j}(X)$ satisfies the conditions (1) $\sim$ (3) in the corollary. The sufficiency has been proved in Corollary 2.4. □

3. Quasi-pseudo-metrizability and pairwise k-semi-stratifiability

In [15], some characterizations on the quasi-metrizability of a bitopological space are given by means of pairwise weak base g-functions, which generalize some metrization theorems on topological spaces. Marín [17] obtained some quasi-pseudo-metrization by using pairwise stratifiable and γ-spaces. In this section, we shall characterize the quasi-pseudo-metrizability of a bitopological space by using pairwise k-semi-stratifiable and γ-spaces.

Theorem 3.1. *If a bitopological space (X, τ_1, τ_2) is pairwise k-semi-stratifiable and (X, τ_i) is a first countable space for $i = 1, 2$, then (X, τ_1, τ_2) is pairwise stratifiable.*

Proof. Let (X, τ_1, τ_2) be a pairwise k-semi-stratifiable space and D_{ij} $(i, j = 1, 2$ and $i \neq j)$ operators on X which satisfy the conditions (1) $\sim$ (3) in Definition 1.3. We show that $A = \bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\tau_i} D_{ij}(n, A)$ for each $A \in \tau_i^c$. It is enough to prove that $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\tau_i} D_{ij}(n, A) \subseteq A$. Let $x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\tau_i} D_{ij}(n, A)$ and $\mathcal{B}_x = \{B_{n,x} : n \in \mathbb{N}\}$ be a decreasing local base of neighborhoods at x in (X, τ_i). Since $x \in \mathrm{cl}_{\tau_i} D_{ij}(n, A)$ for each $n \in \mathbb{N}$, there is $x_n \in B_{n,x} \cap D_{ij}(n, A)$. Then the sequence $\{x_n\}$ τ_i-converges to x. Put $K_m = \{x\} \cup \{x_n : n \geqslant m\}$ for each $m \in \mathbb{N}$. Then K_m is τ_i-compact and $K_m \cap D_{i,j}(n, A) \neq \emptyset$ for each $n \in \mathbb{N}$. By (3) in Definition 1.3, $K_m \cap A \neq \emptyset$. It follows from $A \in \tau_i^c$ that $x \in A$. Therefore, (X, τ_1, τ_2) is pairwise stratifiable. □

Lemma 3.2. *[17, Theorem 4] A bitopological space (X, τ_1, τ_2) is quasi-pseudo-metrizable if and only if it is pairwise stratifiable and each (X, τ_i) is a γ-space for $i = 1, 2$.*

Since every γ-space is first-countable [5], we obtain the following result by Theorem 3.1 and Lemma 3.2.

Theorem 3.3. *A bitopological space (X, τ_1, τ_2) is quasi-pseudo-metrizable if and only if it is pairwise k-semi-stratifiable and (X, τ_i) is a γ-space for each $i = 1, 2$.*

It is well known that each Fréchet and k-semi-stratifiable space is a stratifiable space [16, Theorem 3.4.3], where a space X is said to be Fréchet if, for each $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X. The following question is posed.

Question 3.4. *Is a pairwise k-semi-stratifiable space (X, τ_1, τ_2) a pairwise stratifiable space if (X, τ_i) is a Fréchet space for each $i = 1, 2$?*

Addendum

Question 3.4 was answered affirmatively by Kedian Li and Jiling Cao, see Theorem 4.2 in [13].

Acknowledgements

The authors would like to thank the referee, who gives a series of essential suggestions which deal with terminology, contents, and expression in English of the paper.

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