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The k_R-property in free topological groups

Fucai Lin[a,*], Shou Lin[b], Chuan Liu[c]

[a] School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, PR China
[b] Institute of Mathematics, Ningde Teachers' College, Ningde, Fujian 352100, PR China
[c] Department of Mathematics, Ohio University Zanesville Campus, Zanesville, OH 43701, USA

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Abstract

A space X is called a k_R-space, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f to each compact subset is continuous. In this paper, we mainly discuss the k_R-property in the free topological groups, and generalize some well-known results of K. Yamada.

Keywords: k_R-space; k-space; Stratifiable space; Lašnev space; k-network; Free topological group

1. Introduction

Recall that X is called a *k-space*, if the necessary and sufficient condition for a subset A of X to be closed is that $A \cap C$ is closed for every compact subset C. It is well-known that the k-property which generalizes metrizability has been studied intensively by topologists and analysts. A space X is called a k_R-space, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f to each compact subset is continuous. Clearly every Tychonoff k-space is a *k_R-space*. The converse is false. Indeed, for any a non-measurable cardinal κ the power $\mathbb{R}^\kappa$ is a k_R-space but not a k-space,

* Corresponding author.
E-mail addresses: linfucai2008@aliyun.com, linfucai@mnnu.edu.cn (F. Lin), shoulin60@163.com (S. Lin), liuc1@ohio.edu (C. Liu).

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see [17, Theorem 5.6] and [11, Problem 7.J(b)]. The k_R-property has been widely used in the study of topology, analysis and category, see [3–6,12,13,16].

The results of our research will be presented in two separate papers. In the paper [15], we mainly extend some well-known results in k-spaces to k_R-spaces, and then seek some applications in the study of free Abelian topological groups. In the current paper, we shall detect the k_R-property in free topological groups and extend some results of K. Yamada.

The paper is organized as follows. In Section 2, we introduce the necessary notation and terminologies which are used for the rest of the paper. In Section 3, we investigate the k_R-property in free topological groups, and generalize some results of K. Yamada. In Section 4, we pose some interesting questions about k_R-spaces in the class of free topological groups which are still unknown to us.

2. Preliminaries

In this section, we introduce the necessary notation and terminologies. Throughout this paper, all topological spaces are assumed to be Tychonoff, unless otherwise is explicitly stated. First of all, let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. For a space X, we always denote the set of all the non-isolated points by $\mathrm{NI}(X)$. For undefined notation and terminologies, the reader may refer to [2,8,9] and [14].

Let X be a topological space and $A \subseteq X$ be a subset of X. The *closure* of A in X is denoted by $\overline{A}$. Moreover, A is called *bounded* if every continuous real-valued function f defined on A is bounded. The space X is called a *k-space* provided that a subset $C \subseteq X$ is closed in X if $C \cap K$ is closed in K for each compact subset K of X. A space X is called a k_R-space, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f to each compact subset is continuous. Note that every Tychonoff k-space is a k_R-space. A subset P of X is called a *sequential neighborhood* of $x \in X$, if each sequence converging to x is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. A subset F of X is called *sequentially closed* if $X \setminus F$ is sequentially open. The space X is called a *sequential space* if each sequentially open subset of X is open. The space X is said to be *Fréchet–Urysohn* if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and $\{x_n : n \in \mathbb{N}\} \subset A$.

Definition 2.1 (*[3]*)**.** Let X be a topological space.

• A subset U of X is called *$\mathbb{R}$-open* if for each point $x \in U$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$. It is obvious that each $\mathbb{R}$-open set is open. The converse is true for the open subsets of Tychonoff spaces.

• A subset U of X is called a *functional neighborhood* of a set $A \subset X$ if there is a continuous function $f : X \to [0, 1]$ such that $f(A) \subset \{1\}$ and $f(X \setminus U) \subset \{0\}$. If X is normal, then each neighborhood of a closed subset $A \subset X$ is functional.

Definition 2.2. Let λ be a cardinal. An indexed family $\{X_\alpha\}_{\alpha\in\lambda}$ of subsets of a topological space X is called

• *point-countable* if for any point $x \in X$ the set $\{\alpha \in \lambda : x \in X_\alpha\}$ is countable;

• *compact-countable* if for any compact subset K in X the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is countable;

• *locally finite* if any point $x \in X$ has a neighborhood $O_x \subset X$ such that the set $\{\alpha \in \lambda : O_x \cap X_\alpha \neq \emptyset\}$ is finite;

• *compact-finite* in X if for each compact subset $K \subset X$ the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is finite;

• *strongly compact-finite* [3] in X if each set X_α has an $\mathbb{R}$-open neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite;

• *strictly compact-finite* [3] in X if each set X_α has a functional neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite.

Definition 2.3 (*[3]*). Let X be a topological space and λ be a cardinal. An indexed family $\{F_\alpha\}_{\alpha\in\lambda}$ of subsets of a topological space X is called a *fan* (more precisely, a λ-fan) in X if this family is compact-finite but not locally finite in X. A fan $\{F_\alpha\}_{\alpha\in\lambda}$ is called *strong* (resp. *strict*) if each set F_α has a $\mathbb{R}$-open neighborhood (resp. functional neighborhood) $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite in X.

If all the sets F_α of a λ-fan $\{F_\alpha\}_{\alpha\in\lambda}$ belong to some fixed family $\mathscr{F}$ of subsets of X, then the fan will be called an *$\mathscr{F}^\lambda$-fan*. In particular, if each F_α is closed in X, then the fan will be called a *Cld^λ-fan*.

Clearly, we have the following implications:

$$\text{strict fan} \Rightarrow \text{strong fan} \Rightarrow \text{fan}.$$

Let $\mathscr{P}$ be a family of subsets of a space X. Then, $\mathscr{P}$ is called a *k-network* if for every compact subset K of X and an arbitrary open set U containing K in X there is a finite subfamily $\mathscr{P}' \subseteq \mathscr{P}$ such that $K \subseteq \bigcup \mathscr{P}' \subseteq U$. Recall that a space X is an *$\aleph$-space* (resp. *$\aleph_0$-space*) if X has a σ-locally finite (resp. countable) k-network. Recall that a space X is said to be *Lašnev* if it is the continuous closed image of some metric space.

Definition 2.4 (*[7]*). A topological space X is a *stratifiable space* if for each open subset U in X, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

(a) $\overline{U_n} \subset U$;

(b) $\bigcup_{n=1}^{\infty} U_n = U$;

(c) $U_n \subset V_n$ whenever $U \subset V$.

Note: Each Lašnev space is stratifiable [9].

Let X be a non-empty space. Throughout this paper, $X^{-1} = \{x^{-1} : x \in X\}$, which is just a copy of X. For every $n \in \mathbb{N}$, $F_n(X)$ denotes the subspace of $F(X)$ that consists of all the words of reduced length at most n with respect to the free basis X. Let e be the neutral element of $F(X)$, that is, the empty word. For every $n \in \mathbb{N}$, an element $x_1x_2\cdots x_n$ is also called a *form* for $(x_1, x_2, \ldots, x_n) \in (X \bigoplus X^{-1} \bigoplus \{e\})^n$. The word g is called *reduced* if it does not contain e or any pair of consecutive symbol of the form xx^{-1}. It follows that if the word g is reduced and non-empty, then it is different from the neutral element e of $F(X)$. In particular, each element $g \in F(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$, where $n \geq 1$, $\epsilon_1 \in \{-1, 1\}$, $x_i \in X$, and the *support* of $g = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ is defined as $\mathrm{supp}(g) = \{x_1, \ldots, x_n\}$. Given a subset K of $F(X)$, we put $\mathrm{supp}(K) = \bigcup_{g\in K} \mathrm{supp}(g)$. For every $n \in \mathbb{N}$, let

$$i_n : (X \bigoplus X^{-1} \bigoplus \{e\})^n \to F_n(X)$$

be the natural mapping defined by

$$i_n(x_1, x_2, \ldots, x_n) = x_1x_2\cdots x_n$$

for each $(x_1, x_2, \ldots, x_n) \in (X \bigoplus -X \bigoplus \{0\})^n$.

3. The k_R -property in free topological groups

In this section, we investigate the k_R-property in free topological groups, and generalize some results of K. Yamada. Recently, T. Banakh in [3] proved that $F(X)$ is a k-space if $F(X)$ is a k_R-space for a Lašnev space X. Indeed, he obtained this result in the class of weaker spaces. However, he did not discuss the following question:

Question 3.1. *Let X be a space. For some $n \in \omega$, if $F_n(X)$ is a k_R-space, is $F_n(X)$ a k-space?*

First, we present Theorem 3.3, which complements the result of T. Banakh.

Lemma 3.2. *Let $F(X)$ be a k_R-space. If each $F_n(X)$ is a normal k-space, then $F(X)$ is a k-space.*

Proof. It is well-known that each compact subset of $F(X)$ is contained in some $F_n(X)$ [2, Corollary 7.4.4]. Hence it follows from [12, Lemma 2] that $F(X)$ is a k-space. □

Theorem 3.3. *Let X be a paracompact σ-space. Then $F(X)$ is a k_R-space and each $F_n(X)$ is a k-space if and only if $F(X)$ is a k-space.*

Proof. Since X is a paracompact σ-space, it follows from [2, Theorem 7.6.7] that $F(X)$ is also a paracompact σ-space, hence each $F_n(X)$ is normal. Now apply Lemma 3.2 to conclude the proof. □

Next, we shall prove that for an arbitrary metrizable space X, the k_R-property of $F_8(X)$ implies that $F(X)$ is a k-space, Theorem 3.6. We first prove two technical propositions. To prove them, we need the description of a neighborhood base of e in $F(X)$ obtained in [18]. Let

$$H_0(X) = \left\{ h = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{2n}^{\varepsilon_{2n}} \in F(X) : \sum_{i=1}^{2n} \varepsilon_i = 0, x_i \in X \text{ for } i \in \{1, 2, \ldots, n\}, n \in \mathbb{N} \right\}.$$

Obviously, the subset $H_0(X)$ is a clopen normal subgroup of $F(X)$. It is easy to see that each $h \in H_0(X)$ can be represented as

$$h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} g_2 x_2^{\varepsilon_2} y_2^{-\varepsilon_1} g_2^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_1} g_n^{-1},$$

for some $n \in \mathbb{N}$, where $x_i, y_i \in X$, $\varepsilon_i = \pm 1$ and $g_i \in F(X)$ for $i \in \{1, 2, \ldots, n\}$. Let $P(X)$ be the set of all continuous pseudometrics on a space X. Then take an arbitrary $r = \{\rho_g : g \in F(X)\} \in P(X)^{F(X)}$. Let

$$p_r(h) = \inf \left\{ \sum_{i=1}^{n} \rho_{g_i}(x_i, y_i) : h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} g_2 x_2^{\varepsilon_2} y_2^{-\varepsilon_1} g_2^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_1} g_n^{-1}, n \in \mathbb{N} \right\}$$

for each $h \in H_0(X)$. In [18], Uspenskii proved that

(a) ρ_r is continuous on $H_0(X)$ and

(b) $\{\{h \in H_0(X) : p_r(h) < \delta\} : r \in P(X)^{F(X)}, \delta > 0\}$ is a neighborhood base at e in $F(X)$. Moreover, $p_r(e) = 0$ for each $r \in P(X)^{F(X)}$.

Proposition 3.4. *For a stratifiable k-space X, if $F_8(X)$ is a k_R-space, then X is separable or discrete.*

Proof. Assume to the contrary that X is neither separable nor discrete. Then X contains a closed subspace $Y = C \bigoplus D$, where $C = \{x_n : n \in \omega\} \cup \{x\}$ is a convergent sequence with its limit point $\{x\}$ and $D = \{d_\alpha : \alpha \in \omega_1\}$ is a discrete closed subset of X. Since D is a discrete closed subset of X, we choose a discrete family $\{O_\alpha\}_{\alpha\in\omega_1}$ of open subsets such that $d_\alpha \in O_\alpha$ for each $\alpha \in \omega_1$. We may assume that $x_n \neq x_m$ for arbitrary $n \neq m$ and $C \cap \bigcup_{\alpha\in\omega_1} O_\alpha = \emptyset$. Since X is stratifiable and Y is closed in X, it follows from [18] that $F(Y)$ is homeomorphic to a closed subgroup of $F(X)$. Hence $F_8(Y)$ is closed subspace of $F_8(X)$. Next we shall prove that $F_8(X)$ contains a strict Cld^ω-fan, a contradiction. Indeed, since $F_8(X)$ is normal, it suffices to construct a strong Cld^ω-fan in $F_8(X)$.

For each $\alpha \in \omega_1$ choose a function $f_\alpha : \omega_1 \to \omega$ such that $f_\alpha|_\alpha : \alpha \to \omega$ is a bijection. For each $n \in \omega$, let

$$F_n = \{d_\beta^{-1}x^{-1}x_m d_\beta d_\alpha x_n x^{-1} d_\alpha^{-1} : f_\alpha(\beta) = n, \alpha, \beta \in \omega_1, m \leq n\}.$$

We claim that the family $\{F_n : n \in \omega\}$ is a strong Cld^ω-fan in $F_8(X)$. We divide the proof into the following three statements.

(1) For each $n \in \omega$, the set F_n is closed in $F_8(X)$.

Fix an arbitrary $n \in \omega$. It suffices to prove that the set F_n is closed in $F_8(Y)$. Let $Z = \mathrm{supp} F_n$. It is obvious that Z is a closed discrete subspace of Y. It follows from [18] that $F(Z)$ is topologically isomorphic to a closed subgroup of $F(Y)$, and thus $F_8(Z)$ is a closed subspace of $F_8(Y)$. Since $F(Z)$ is discrete and $F_n \subset F_8(Z)$, the set F_n is closed in $F_8(Y)$ (and thus closed in $F(X)$).

(2) The family $\{F_n : n \in \omega\}$ is strong compact-finite in $F_8(X)$.

By induction, choose two families of open neighborhoods $\{W_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ in X that satisfy the following conditions:

(a) for each $n \in \omega$, $x_n \in W_n$;

(b) for each $n \in \omega$, $x \in V_n$ and $V_{n+1} \subset V_n$;

(c) for each $n \in \omega$, we have $W_n \cap (C \cup D) = \{x_n\}$, $V_n \cap (D \cup W_n) = \emptyset$ and $V_n \cap C \subset C_n$, where $C_n = \{x_m : m > n\} \cup \{x\}$;

(d) $V_1 \cap \bigcup_{\alpha\in\omega_1} O_\alpha = \emptyset$ and $W_n \cap \bigcup_{\alpha\in\omega_1} O_\alpha = \emptyset$ for each $n \in \omega$.

For each $n \in \omega$, let

$$U_n = \bigcup\{O_\beta^{-1} V_n^{-1} W_m O_\beta O_\alpha W_n V_n^{-1} O_\alpha^{-1} : f_\alpha(\beta) = n, \alpha, \beta \in \omega_1, m \leq n\}.$$

Obviously, each U_n is contained in $F_8(X) \setminus F_7(X)$, and since $F_8(X) \setminus F_7(X)$ is open in $F_8(X)$, it follows from [2, Corollary 7.1.19] that each U_n is open in $F_8(X)$. We claim that the family $\{U_n : n \in \omega\}$ is compact-finite in $F_8(X)$. If not, then there exist a compact subset K in $F_8(X)$ and an increasing sequence $\{n_k\}$ such that $K \cap U_{n_k} \neq \emptyset$ for each $k \in \omega$. Since X is stratifiable, $F(X)$ is paracompact, hence the closure of the set $\mathrm{supp}(K)$ is compact in X. However, for each $k \in \omega$, since $K \cap U_{n_k} \neq \emptyset$, there exist $m_k \in \omega$, $\alpha_k \in \omega_1$ and $\beta_k \in \omega_1$ such that $f_{\alpha_k}(\beta_k) = n_k$ and

$$K \cap O_{\beta_k}^{-1} V_{n_k}^{-1} W_{m_k} O_{\beta_k} O_{\alpha_k} W_{n_k} V_{n_k}^{-1} O_{\alpha_k}^{-1} \neq \emptyset,$$

hence $\mathrm{supp}(K) \cap O_{\alpha_k} \neq \emptyset$ and $\mathrm{supp}(K) \cap O_{\beta_k} \neq \emptyset$. Therefore, the set $\mathrm{supp}(K)$ intersects each element of the family $\{O_{\alpha_k}, O_{\beta_k} : k \in \omega\}$. Since the set $\{n_k : k \in \omega\}$ is infinite and $f_{\alpha_k}(\beta_k) = n_k$ for each $k \in \omega$, the set $\{\alpha_k, \beta_k : k \in \omega\}$ is infinite. Then $\mathrm{supp}(K)$ intersects infinitely many O_α's, which is a contradiction with the compactness of $\mathrm{supp}(K)$. Therefore, the family $\{U_n : n \in \omega\}$ is compact-finite in $F_8(X)$. Therefore, the family $\{F_n : n \in \omega\}$ is strong compact-finite in $F_8(X)$.

(3) The family $\{F_n : n \in \omega\}$ is not locally finite at the point e in $F_8(Y)$ (and thus not locally finite in $F_8(X)$).

Indeed, it suffices to prove that $e \in \overline{\bigcup_{n\in\omega} F_n} \setminus \bigcup_{n\in\omega} F_n$ in $F_8(Y)$. For any $\delta > 0$ and $r = \{\rho_g : g \in F(Y)\}$, we shall prove that

$$\{h \in F_8(Y) : p_r(h) < \delta\} \cap \bigcup_{n\in\omega} F_n \neq \emptyset.$$

Since the sequence $\{x_n\}$ converges to x and ρ_{d_α} and $\rho_{d_\alpha^{-1}}$ are continuous pseudometrics on Y for each $\alpha \in \omega_1$, there is $n(\alpha) \in \omega$ such that $\rho_{d_\alpha}(x_n, x) < \frac{\delta}{2}$ and $\rho_{d_\alpha^{-1}}(x_n, x) < \frac{\delta}{2}$ for each $n \geq n(\alpha)$. Therefore, there are $n_0 \in \omega$ and an uncountable set $A \subset \omega_1$ such that $\rho_{d_\alpha}(x_n, x) < \frac{\delta}{2}$ and $\rho_{d_\alpha^{-1}}(x_n, x) < \frac{\delta}{2}$ for each $n \geq n_0$ and $\alpha \in A$. Choose $\alpha \in A$ that has infinitely many predecessors in A. Since $f_\alpha(\alpha \cap A)$ is an infinite set, there exist $m > n_0$ and $\beta \in \alpha \cap A$ such that $f_\alpha(\beta) = m$. Then the word

$$g = d_\beta^{-1} x^{-1} x_{n_0} d_\beta d_\alpha x_{f_\alpha(\beta)} x^{-1} d_\alpha^{-1} \in F_{f_\alpha(\beta)} = F_m.$$

Furthermore, we have

$$p_r(g) = p_r(d_\beta^{-1} x^{-1} x_{n_0} d_\beta d_\alpha x_{f_\alpha(\beta)} x^{-1} d_\alpha^{-1}) \leq \rho_{d_\beta^{-1}}(x_{n_0}, x) + \rho_{d_\alpha}(x_m, x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence $e \in \overline{\bigcup_{n\in\omega} F_n}$. Hence the family $\{F_n : n \in \omega\}$ is not locally finite at the point e in $F_8(Y)$. □

Proposition 3.5. *For a metrizable space X, if $F_8(X)$ is a k_R-space, then X is locally compact.*

Proof. Assume to the contrary that X is not locally compact. Then there exists a closed hedgehog subspace

$$J = \{x\} \cup \left(\bigcup_{n\in\omega} X_n\right) \cup \{z_n : n \in \omega\}$$

such that

(1) $X_n = \{y_n\} \cup \{x_{n,j} : j \in \omega\}$ is a closed discrete subset of J for each $n \in \omega$;
(2) $\{z_n : n \in \omega\}$ is a closed discrete subset of J; and
(3) $\{\{x\} \cup \bigcup_{n\geq k} X_n : k \in \omega\}$ is a neighborhood base of x in J.

By Proposition 3.4, X is separable. Next we shall prove that $F_8(X)$ contains a strict Cld^ω-fan, which is a contradiction with $F_8(X)$ being a k_R-space. Since $F_8(X)$ is an $\aleph_0$-space by [1, Theorem 4.1], the subspace $F_8(X)$ is normal, hence it suffices to prove that $F_8(X)$ contains a strong Cld^ω-fan. Furthermore, it follows from [3, Proposition 2.9.2] that each compact-finite family of subsets of X is strongly compact-finite, hence it suffices to prove that $F_8(X)$ contains a Cld^ω-fan. For any $n, j \in \omega$, put

$$E_{n,j} = \{z_n^{-1} y_j^{-1} x z_n x_{n,j}\}.$$

It is obvious that each $E_{n,j}$ is closed. Furthermore, it follows from the proof of [20, Proposition 2.1] that $x \in \overline{\bigcup_{n,j\in\omega} E_{n,j}} \setminus \bigcup_{n,j\in\omega} E_{n,j}$, and thus the family $\{E_{n,j} : n, j \in \omega\}$ is not locally finite at the point x. Next we claim that the family $\{E_{n,j} : n, j \in \omega\}$ is compact-finite.

If not, then there exist a compact subset K and two sequences $\{n_i\}$ and $\{j_i\}$ such that $K \cap E_{n_i,j_i} \neq \emptyset$. The closure of the set $\mathrm{supp}(K)$ is compact since $F_8(X)$ is paracompact. Since the family $\{E_{n,j} : n, j \in \omega\}$ is pairwise disjoint, one of the sequences $\{n_i\}$ and $\{j_i\}$ is an infinite set. If $\{n_i\}$ is an infinite set, then the closed discrete set $\{z_{n_i} : i \in \omega\}$ is contained in $\mathrm{supp}(K)$, which is a contradiction since $\overline{\mathrm{supp}(K)}$ is compact. If $\{j_i\}$ is an infinite set and $\{n_i\}$ is a finite set, then there exists $N \in \omega$ such that $\{x_{n_i,j_i} : i \in \mathbb{N}\} \subset \bigcup_{j<N} X_j$. Obviously, the closed discrete set $\{x_{n_i,j_i} : i \in \mathbb{N}\}$ is an infinite set and contained in $\operatorname{supp}(K)$, which is a contradiction. □

Now we can prove one of the main theorems in this paper.

Theorem 3.6. *For a metrizable space X, the following are equivalent:*

(1) *$F(X)$ is a k-space;*
(2) *$F_8(X)$ is a k-space;*
(3) *$F_8(X)$ is a k_R-space;*
(4) *the space X is locally compact separable or discrete.*

Proof. The equivalence of (1) and (4) was proved in [20]. It is obvious that (2) ⇒ (3). By Propositions 3.4 and 3.5, we have (3) ⇒ (4). □

By Theorem 3.6, it is natural to ask the following question:

Question 3.7. *Let X be a metrizable space. If $F_n(X)$ is a k_R-space for some $n \in \{4, 5, 6, 7\}$, then is $F_8(X)$ a k_R-space?*

Note For each $n \in \{2, 3\}$, the answer to the above question is negative. Indeed, for an arbitrary metrizable space X, since i_2 is a closed mapping, $F_2(X)$ is a Fréchet–Urysohn space (and thus a k-space). For $n = 3$, we have the following Theorem 3.9. However, for each $n \in \{4, 5, 6, 7\}$, the above question is still unknown to us.

Proposition 3.8. *For a metrizable space X, if $F_3(X)$ is a k_R-space, then X is locally compact or NI(X) is compact.*

Proof. Assume to the contrary that neither X is locally compact nor the set of all non-isolated points of X is compact. Then X contains a closed subspace

$$Y = \{x\} \cup \bigcup_{n\in\omega} X_n \oplus \bigoplus_{n\in\omega} C_n,$$

where for every $n \in \omega$

$X_n = \{x_{n,i} : i \in \omega\}$ is a closed discrete subset of X,

$\{\{x\} \cup \bigcup_{m\geq n} X_m : n \in \omega\}$ is a neighborhood base at x in Y,

$C_n = \{c_{n,i} : i \in \omega\} \cup \{c_n\}$ is a convergent sequence with its limit c_n, and

C_n is contained in the open subset U_n of X such that the family $\{U_n : n \in \omega\}$ is discrete in X and $(\{x\} \cup \bigcup_{m\in\omega} X_m) \cap \overline{\bigcup_{m\in\omega} U_m} = \emptyset$.

In order to obtain a contradiction, we shall construct a strict Cld$^\omega$-fan in $F_3(X)$. For any $n, i \in \omega$, choose an open neighborhood O_n^i of the point $x_{n,i}$ in X such that the family $\{O_n^i : i \in \omega\}$ is discrete, $O_n^i \cap \overline{\bigcup_{m\in\omega} U_m} = \emptyset$ and $O_n^i \cap (\{x\} \cup \bigcup_{n\in\omega} X_n) = \{x_{n,i}\}$.

For each $n, i \in \omega$, let

$$E(n, i) = \{g_{n,i} = c_n c_{n,i}^{-1} x_{n,i}\}$$

and

$$U(n, i) = V_n^i (W_n^i)^{-1} O_n^i,$$

where V_n^i and W_n^i are two arbitrary open neighborhoods of c_n and $c_{n,i}$ in X respectively such that $V_n^i \cup W_n^i \subset U_n$ and $V_n^i \cap W_n^i = \emptyset$. Obviously, each $E(n, i)$ is closed and it follows from

[2, Corollary 7.1.19] that $U(n,i)$ is an open neighborhood of $E(n,i)$ for each $n, i \in \omega$. In [20], the author has proved that $x \in \overline{\bigcup_{n,i} E(n,i)} \setminus \bigcup_{n,i} E(n,i)$, hence the family $\{E(n,i) : n, i \in \omega\}$ is not locally finite in $F_3(X)$. To complete the proof, it suffices to prove that the family $\{U(n,i) : n, i \in \omega\}$ is compact-finite in $F_3(X)$. If not, then there exists a compact subset K and two sequences $\{n_j\}$ and $\{i_j\}$ such that $K \cap U(n_j, i_j) \neq \emptyset$. Similar to the proof of Proposition 3.5, we can now obtain a contradiction. □

Theorem 3.9. *For a metrizable space X, the following are equivalent:*

(1) *$F_3(X)$ is a k-space;*
(2) *$F_3(X)$ is a k_R-space;*
(3) *the space X is locally compact or $NI(X)$ is compact.*

Proof. The equivalence of (1) and (3) was proved in [20]. The implication of (1) $\Rightarrow$ (2) is obvious. By Proposition 3.8, we have (2) $\Rightarrow$ (3). □

The following theorem was proved in [15].

Theorem 3.10. [15] *Let X be a stratifiable space such that X^2 is a k_R-space. If X satisfies one of the following conditions, then either X is metrizable or X is the topological sum of k_ω-subspaces.*

(1) *X is a k-space with a compact-countable k-network;*
(2) *X is a Fréchet–Urysohn space with a point-countable k-network.*

Since $F_2(X)$ contains a closed copy of $X \times X$, it follows from Theorem 3.10 that we have the following theorem.

Theorem 3.11. *Let X be a stratifiable k-space with a compact-countable k-network. Then the following are equivalent:*

(1) *$F_2(X)$ is a k-space;*
(2) *$F_2(X)$ is a k_R-space;*
(3) *either X is metrizable or X is the topological sum of k_ω-subspaces.*

The following proposition shows that we cannot replace "$F_3(X)$" with "$F_4(X)$" in Theorem 3.9. First, we recall a special space. Let

$$M_3 = \bigoplus\{C_\alpha : \alpha < \omega_1\},$$

where $C_\alpha = \{c(\alpha, n) : n \in \mathbb{N}\} \cup \{c_\alpha\}$ with $c(\alpha, n) \to c_\alpha$ as $n \to \infty$ for each $\alpha \in \omega_1$.

Proposition 3.12. *The subspace $F_4(M_3)$ is not a k_R-space.*

Proof. It suffices to prove that $F_4(M_3)$ contains a strict Cld^ω-fan. It follows from [10, Theorem 20.2] that we can find two families $\mathscr{A} = \{A_\alpha : \alpha \in \omega_1\}$ and $\mathscr{B} = \{B_\alpha : \alpha \in \omega_1\}$ of infinite subsets of ω such that

(a) $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta \in \omega_1$;
(b) for no $A \subset \omega$, all the sets $A_\alpha \setminus A$ and $B_\alpha \cap A$, $\alpha \in \omega_1$ are finite.

For each $n \in \omega$, put

$$X_n = \{c(\alpha, n)c_\alpha^{-1}c(\beta, n)c_\beta^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.$$

It suffices to prove the following three statements.

(1) The family $\{X_n\}$ is strictly compact-finite in $F_4(M_3)$.

Since M_3 is a Lašnev space, it follows from [2, Theorem 7.6.7] that $F(M_3)$ is also a paracompact σ-space, hence $F_4(M_3)$ is paracompact (and thus normal). Hence it suffices to prove that the family $\{X_n\}$ is strongly compact-finite in $F_4(M_3)$. For each $\alpha \in \omega_1$ and $n \in \omega$, let $C_\alpha^n = C_\alpha \setminus \{c(\alpha, m) : m \leq n\}$, and put

$$U_n = \{c(\alpha, n)x^{-1}c(\beta, n)y^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1, x \in C_\alpha^n, y \in C_\beta^n\}.$$

Obviously, each $X_n \subset F_4(M_3) \setminus F_3(M_3)$. Since $F_4(M_3) \setminus F_3(M_3)$ is open in $F_4(M_3)$, it follows from [2, Corollary 7.1.19] that each U_n is open in $F_4(M_3)$. We claim that the family $\{U_n\}$ is compact-finite in $F_4(M_3)$. If not, then there exist a compact subset K in $F_4(M_3)$ and a subsequence $\{n_k\}$ of ω such that $K \cap U_{n_k} \neq \emptyset$ for each $k \in \omega$. For each $k \in \omega$, choose an arbitrary point

$$z_k = c(\alpha_k, n_k)x_k^{-1}c(\beta_k, n_k)y_k^{-1} \in K \cap U_{n_k},$$

where $x_k \in C_{\alpha_k}^{n_k}$ and $y_k \in C_{\beta_k}^{n_k}$. Since $F_4(M_3)$ is paracompact, it follows from [1] that the closure of the set $\mathrm{supp}(K)$ is compact in M_3. Therefore, there exists $N \in \omega$ such that

$$\mathrm{supp}(K) \cap \bigcup\{C_\alpha : \alpha \in \omega_1 \setminus \{\alpha_i \in \omega_1 : i \leq N\}\} = \emptyset,$$

that is, $\mathrm{supp}(K) \subset \bigcup_{\alpha \in \{\gamma_i \in \omega_1 : i \leq N\}} C_\alpha$. Since each $z_k \in K$, there exists

$$\alpha_k, \beta_k \in \{\alpha_i \in \omega_1 : i \leq N\}$$

such that $A_{\alpha_k} \cap B_{\beta_k}$ is an infinite set, which is a contradiction since $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta < \omega_1$.

(2) Each X_n is closed in $F_4(M_3)$.

Fix an arbitrary $n \in \omega$. We shall prove that X_n is closed in $F_4(M_3)$. Let $Z = \mathrm{supp}(X_n)$. Then Z is a closed discrete subset of M_3. Since M_3 is metrizable, it follows from [18] that $F(Z)$ is homeomorphic to a closed subgroup of $F(M_3)$, hence $F_4(Z)$ is a closed subspace of $F_4(M_3)$. Since $F(Z)$ is discrete and $X_n \subset F_4(Z)$, the set X_n is closed in $F_4(Z)$ (and thus closed in $F_4(M_3)$).

(3) The family $\{X_n\}$ is not locally finite at the point e in $F_4(M_3)$.

Indeed, it suffices to prove that $e \in \overline{\bigcup_{n\in\omega} X_n} \setminus \bigcup_{n\in\omega} X_n$. For any $\delta > 0$ and $r = \{\rho_g : g \in F(M_3)\}$, we shall prove that

$$\{h \in F_4(M_3) : p_r(h) < \delta\} \cap \bigcup_{n\in\omega} X_n \neq \emptyset$$

since ρ_e is continuous, we can choose a function $f : \omega_1 \to \omega$ such that $\rho_e(c(\alpha, n), c_\alpha) < \frac{\delta}{2}$ for any $\alpha \in \omega_1$ and $n \geq f(\alpha)$. For each $\alpha < \omega_1$, put $A'_\alpha = \{n \in A_\alpha : n \geq f(\alpha)\}$ and $B'_\alpha = \{n \in B_\alpha : n \geq f(\alpha)\}$. By the condition (b) of the families $\mathscr{A}$ and $\mathscr{B}$, it is easy to see that there exist $\alpha, \beta \in \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. So, choose $n \in A'_\alpha \cap B'_\beta$. Then $\rho_e(c(\alpha, n), c_\alpha) < \frac{\delta}{2}$ and $\rho_e(c(\beta, n), c_\beta) < \frac{\delta}{2}$. Let $z = c(\alpha, n)c_\alpha^{-1}c(\beta, n)c_\beta^{-1}$. Then $z \in X_n$ and

$$p_r(z) \leq \rho_e(c(\alpha, n), c_\alpha) + \rho_e(c(\beta, n), c_\beta) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

hence $z \in \{h \in F_4(M_3) : p_r(h) < \delta\} \cap \bigcup_{n\in\omega} X_n$. $\square$

Theorem 3.13. *Let X be a metrizable space. If $F_4(X)$ is a k_R-space, then $NI(X)$ is separable.*

Proof. If not, then X contains a closed copy of M_3. Use the same notation as in Proposition 3.12. Since X is metrizable, there exists a discrete family $\{U_\alpha\}_{\alpha\in\omega_1}$ of open subsets in X such that $C_\alpha \subset U_\alpha$ for each $\alpha \in \omega_1$. For arbitrary $(\alpha, n) \in \omega_1 \times \omega$, choose open neighborhoods $V_{\alpha,n}$ and $W_{\alpha,n}$ of the point $c(\alpha, n)$ and c_α in X respectively such that $V_{\alpha,n} \cap W_{\alpha,n} = \emptyset$ and $V_{\alpha,n} \cup W_{\alpha,n} \subset U_\alpha$. For each $n \in \mathbb{N}$, put

$$O_n = \bigcup\{V_{\alpha,n} W_{\alpha,n}^{-1} V_{\beta,n} W_{\beta,n}^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.$$

Similar to the proof of Proposition 3.12, we can prove that the family $\{O_n\}_{n\in\omega}$ of open subsets is compact-finite in $F_4(X)$, hence the family $\{X_n\}_{n\in\omega}$ is a strict Cld^ω-fan in $F_4(X)$, which is a contradiction. □

4. Open questions

In this section, we pose some interesting questions about k_R-spaces in the class of free topological groups, which are still unknown to us.

By Theorems 3.6, 3.9 and 3.11, it is natural to pose the following question.

Question 4.1. *Let X be a metrizable space. For each $n \in \{4, 5, 6, 7\}$, if $F_n(X)$ is a k_R-space, is $F_n(X)$ a k-space?*

In [19], Yamada made the following conjecture:

Yamada's Conjecture: The subspace $F_4(X)$ is Fréchet–Urysohn if the set of all non-isolated points of a metrizable space X is compact.

We do not even know the answer to the following question.

Question 4.2. *Is $F_4(X)$ a k_R-space if the set of all non-isolated points of a metrizable space X is compact?*

In particular, we have the following question.

Question 4.3. *Let $X = C \bigoplus D$, where C is a non-trivial convergent sequence with its limit point and D is an uncountable discrete space. Is $F_4(X)$ a k_R-space?*

In [4], the authors proved that each closed subspace of a stratifiable k_R-space is a k_R-subspace. However, the following two questions are still open.

Question 4.4. *Is each closed subgroup of a k_R-free topological group k_R?*

Question 4.5. *Is each subspace $F_n(X)$ of a k_R-free topological group k_R?*

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References

[1] A.V. Arhangel'skiĭ, O.G. Okunev, V.G. Pestov, Free topological groups over metrizable spaces, Topology Appl. 33 (1989) 63–76.

[2] A. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd, Hackensack, NJ, 2008.

[3] T. Banakh, Fans and their applications in general topology, functional analysis and topological algebra, arXiv:1602.04857v2.

[4] T. Banakh, S. Gabriyelyan, On the C_k-stable closure of the class of (separable) metrizable spaces, Monatsh. Math. 180 (2016) 39–64.

[5] J.L. Blasco, On μ-spaces and k_R-spaces, Proc. Amer. Math. Soc. 67 (1) (1977) 179–186.

[6] C.R. Borges, A stratifiable k_R-space which is not a k-space, Proc. Amer. Math. Soc. 81 (1981) 308–310.

[7] J.G. Ceder, Some generalizations of metric spaces, Pacific J. Math. 11 (1961) 105–C125.

[8] R. Engelking, General Topology (Revised and Completed Edition), Heldermann Verlag, Berlin, 1989.

[9] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V. Amsterdam, 1984, pp. 423–501.

[10] W. Just, M. Weese, Discovering modern set theorey II, Grad. Stud. Math. (1997) 18.

[11] J. Kelley, General Topology, Springer-Verlag, New York-Berlin, 1975.

[12] J. Li, On k-spaces and k_R-spaces, Czechoslovak Math. J. 55 (2006) 941–945.

[13] S. Lin, Note on k_R-spaces, Questions Answers Gen. Topology 9 (1991) 227–236.

[14] S. Lin, Point-Countable Covers and Sequence-Covering Mappings, second ed., China Science Press, Beijing, 2015.

[15] F. Lin, S. Lin, C. Liu, The k_R-property of two product of sequential fans and of free Abelian topological groups, arXiv:1610.02522v4.

[16] E. Michael, On k-spaces, k_R-spaces and $k(X)$, Pacific J. Math. 47 (1973) 487–498.

[17] N. Noble, The continuity of functions on Cartesian products, Trans. Amer. Math. Soc. 149 (1970) 187–C198.

[18] V. Uspenskii, Free topological groups of metrizable spaces, Math. USSR Izv. 37 (1970) 657–680.

[19] K. Yamada, Fréchet-Urysohn spaces in free topological groups, Proc. Amer. Math. Soc. 130 (2002) 2461–2469.

[20] K. Yamada, The natural mappings i_n and k-subspaces of free topological groups on metrizable spaces, Topology Appl. 146–147 (2005) 239–251.