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Some Topological Properties of Charming Spaces

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Abstract: In this paper, we mainly discuss the class of charming spaces. First, we show that there exists a charming space such that the Tychonoff product is not a charming space. Then we discuss some properties of charming spaces and give some characterizations of some class of charming spaces. Finally, we show that the Suslin number of an arbitrary charming rectifiable space is countable.

Key words: charming space, (i, j)-structured space, Lindelöf Σ-space, Suslin number, rectifiable space

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1 Introduction

In 1969, Nagami[1] introduced the notion of Σ-spaces, and then the class of Σ-spaces with the Lindelöf property (i.e., the class of Lindelöf Σ-space) quickly attracted the attention of some topologists. From then on, the study of Lindelöf Σ-spaces has become an important part in the functional analysis, topological algebra and descriptive set theory. Tkachuk[2] described detailedly Lindelöf Σ-spaces and made an overview of the recent progress achieved in the study of Lindelöf Σ-spaces. Arhangel'skii[3] has proved if the weight of X does not exceed 2^ω, then any remainders of X in a Hausdorff compactification is a Lindelöf Σ-space.

It is natural to ask if we can find a class of spaces $\mathscr{P}$ such that each remainder of a Hausdorff compactification of arbitrary metrizable space belongs to $\mathscr{P}$. Therefore, Arhangel'skii

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defined charming spaces in [3] and showed the any remainder of paracompact p-space in a Hausdorff compactification is a charming space. Indeed, Arhangel'skii defined many new classes of spaces (that is, (i, j)-structured spaces) which have similar structure with the class of charming spaces, and he said that each of the classes of spaces so defined is worth studying. Therefore, we mainly discuss some topological properties of (i, j)-structured spaces.

2 Preliminaries

All spaces are Tychonoff unless stated otherwise. Readers may refer to [4]–[5] for notations and terminology not explicitly given here.

Definition 2.1 *Let $\mathscr{N}$ be a family of subsets of a space X. Then the family $\mathscr{N}$ is a network of X if every open subset U is the union of some subfamily of $\mathscr{N}$.*

Definition 2.2 *We say that a space is cosmic if it has a countable network.*

Definition 2.3 *Let X be a space. We say that X is a Lindelöf p-space if it is the preimage of a separable metrizable space under a perfect mapping.*

Definition 2.4 *X is a Lindelöf Σ-space if there exists a space Y which maps continuously onto X and perfectly onto a second countable space.*

That is, a Lindelöf Σ-space is the continuous onto image of some Lindelöf p-space. Therefore, a Lindelöf p-space is a Lindelöf Σ-space. It is well-known that the class of Lindelöf Σ-spaces contains the classes of Σ-compact spaces and spaces with a countable network.

Definition 2.5[3] *Let X be a space. If there exists a Lindelöf Σ-subspace Y such that for each open neighborhood U of Y in X we have $X \setminus U$ is also a Lindelöf Σ-subspace, then we say that X is a charming space.*

Definition 2.6[3] *Let $\mathscr{P}$ and $\mathscr{Q}$ be two classes of topological spaces respectively. A space X will be called $(\mathscr{P}, \mathscr{Q})$-structured if there is a subspace Y of X such that $Y \in \mathscr{P}$, and for each open neighborhood U of Y in X, the subspace $X \setminus U$ of X belongs to $\mathscr{Q}$. In this situation, we call Y a $(\mathscr{P}, \mathscr{Q})$-shell of the space X.*

Definition 2.7[3] *Let $\mathscr{P}_0$ be the class of Σ-compact spaces, $\mathscr{P}_1$ be the class of separable metrizable spaces, $\mathscr{P}_2$ be the class of spaces with a countable network, $\mathscr{P}_3$ be the class of Lindelöf p-spaces, $\mathscr{P}_4$ be the class of Lindelöf Σ-spaces and $\mathscr{P}_5$ be the class of compact spaces. Choose some $i, j \in \{0, 1, 2, 3, 4, 5\}$. A space X is called (i, j)-structured if it is $(\mathscr{P}_i, \mathscr{P}_j)$-structured. In particular, a $(4, 4)$-structured space is called a charming space.*

3 The Properties of (i, j)-structured Spaces

In this section, we discuss some properties of (i, j)-structured spaces.

Proposition 3.1 *Each closed subspace of an (i, j)-structured space X is (i, j)-structured, where $i, j \in \{0, 1, 2, 3, 4, 5\}$.*

Proof. Let X be an (i, j)-structured space, A be a closed subspace of X. Since X is an (i, j)-structured space, there is a subspace Y such that Y is a $(\mathscr{P}_i, \mathscr{P}_j)$-shell of the space X. Let $B = A \bigcap Y$. We claim that B is a $(\mathscr{P}_i, \mathscr{P}_j)$-shell of the space A. Obviously, we have $B \in \mathscr{P}_i$. Now it suffices to show that for each open neighborhood V of B in A, the subspace $A \setminus V$ of A belongs to $\mathscr{P}_j$. Indeed, let V be an open neighborhood of B in A. Then there is an open neighborhood V_1 in X such that

$$V = V_1 \bigcap A, \qquad A \setminus V = A \setminus V_1.$$

We have

$$Y \setminus V_1 \subset X \setminus A, \qquad V_1 \bigcap Y \subset V_1,$$

and

$$(Y \setminus V_1) \bigcup (V_1 \bigcap Y) = Y \subset V_1 \bigcup (X \setminus A).$$

It is easy to see that

$$A \setminus V = A \setminus V_1 \subset X \setminus (V_1 \bigcup (X \setminus A)) \in \mathscr{P}_j.$$

The following three propositions are easy exercises.

Proposition 3.2 *Any image of an (i, j)-structured space under a continuous mapping is an (i, j)-structured space, where $i, j \in \{0, 2, 4, 5\}$.*

Proposition 3.3 *Any preimage of an (i, j)-structured space under a perfect mapping is an (i, j)-structured space, where $i, j \in \{0, 3, 4, 5\}$.*

Proposition 3.4 *For $i \in \{0, 2, 4\}$, if $X_j \in \mathscr{P}_i$ for each $j \in \mathbf{N}$ and $X = \bigcup\limits_{j \in \mathbf{N}} X_j$, then $X \in \mathscr{P}_i$.*

Question 3.1 For arbitrary $i, j \in \{1, 3\}$, is any image of an (i, j)-structured space under a continuous mapping an (i, j)-structured space?

Question 3.2 For arbitrary $i, j \in \{1, 2\}$, is any preimage of an (i, j)-structured space under a perfect mapping an (i, j)-structured space?

Proposition 3.5 *Let X be a space and $X = \bigcup\limits_{k \in \omega} X_k$. If each X_k is an (i, j)-structured space, then X is also an (i, j)-structured space, where $i, j \in \{0, 2, 4\}$.*

Proof. Fix $i, j \in \{0, 2, 4\}$. For every $k \in \omega$, since X_k is an (i, j)-structured space, there exists a subspace $Y_k \subset X_k$ such that Y_k is a $(\mathscr{P}_i, \mathscr{P}_j)$-shell of the space X_k. Put $Y = \bigcup\limits_{k\in\omega} Y_k$. We claim that Y is a $(\mathscr{P}_i, \mathscr{P}_j)$-shell of the space X. Indeed, the space Y belongs to $\mathscr{P}_i$ by Proposition 3.4. Moreover, for each open neighborhood U of Y in X, each $X_k \setminus (X_k \bigcap U) \in \mathscr{P}_j$, hence it follows from Proposition 3.4 that

$$X \setminus U = (\bigcup_{k\in\omega} X_k) \setminus U = \bigcup_{k\in\omega} (X_k \setminus U) = \bigcup_{k\in\omega} (X_k \setminus X_k \bigcap U) \in \mathscr{P}_j.$$

Therefore, X is an (i, j)-structured space.

Since the intersection of countably many Lindelöf Σ-subspaces is also Lindelöf Σ, it is natural to pose the following question.

Question 3.3 Let X be a space and $X_k \subset X$ for each $k \in \mathbf{N}$. If each X_k is an (i, j)-structured space, is $\bigcup\limits_{k\in\mathbf{N}} X_k$ an (i, j)-structured space, where $i, j \in \{0, 1, 2, 3, 4, 5\}$?

It is well-known that the product of a countably many Lindelöf Σ-spaces is also a Lindelöf Σ-space. However, the product of two charming spaces may not be a charming space, see Example 3.1.

We know that each discrete space X has a Hausdorff one point Lindelöfication which defined as follows: take an arbitrary point $p \notin X$ and consider the set $Y = X \bigcup \{p\}$, and then let all points of X be open and each neighborhood of p be the form $U \bigcup \{p\}$, where U is open in X and $X \setminus U$ is a countable set.

Example 3.1 There exists a charming space X satisfying the following conditions:

(i) X is not a Lindelöf Σ-space;

(ii) The product of X^2 is not a charming space.

Proof. Let $X = \{\infty\} \bigcup D$ be the one-point Lindelöfication of an uncountable discrete space, where D is an uncountable discrete space. Tkachuk[2] has proved that X is not a Lindelöf Σ-space. Obviously, the subspace $\{\infty\}$ is a Lindelöf Σ-space, and for each open neighborhood V of $\{\infty\}$ in X we have $D \setminus V$ is Lindelöf, then $D \setminus V$ is separable and metrizable. Thus $D \setminus V$ is a Lindelöf Σ-space. Therefore, X is a charming space.

Now, we shall show that X^2 is not a charming space. Assume that X^2 is a charming space. Then there exists a Lindelöf Σ-subspace $L \subset X^2$ such that, for each open neighborhood U of L, $X^2 \setminus U$ is a Lindelöf Σ-subspace. It is easy to see that $(\infty, \infty) \in L$. Since each compact subset of X^2 is finite, L must be a countable set by Theorem 5 in [2]. Then there exists a point $x_0 \in D$ such that $(\{x_0\} \times X) \bigcap L = \emptyset$ and $(X \times \{x_0\}) \bigcap L = \emptyset$. It is easy to see that $((X \setminus \{x_0\}) \times (X \setminus \{x_0\}))$ is an open neighborhood of L in X^2. Put $V = X \setminus \{x_0\}$. Then

$$X^2 \setminus (V \times V) = \{(x_0, x_0)\} \bigcup (\{x_0\} \times V) \bigcup (V \times \{x_0\})$$

is a Lindelöf Σ-space. Since $\{x_0\} \times V$ is closed in $X^2 \setminus (V \times V)$, we see that $\{x_0\} \times V$ is a Lindelöf Σ-space. However, V is homeomorphic to X, which is a contradiction. Therefore, X^2 is not a charming space.

Remark 3.1 (1) Tkachuk[2] proved that if X is a Lindelöf Σ-space and every compact subset of X is finite then X is countable. From Example 3.1, we know that we can not replace "Lindelöf Σ-spaces" by "charming spaces".

(2) The product of a family of finitely many charming spaces need not be a charming space.

(3) The space X in Example 3.1 is (i, j)-structured for $i, j \in \{0, 1, 2, 3, 4\}$. Therefore, the product of a family of finitely many (i, j)-structured spaces need not be an (i, j)-structured space.

It is easy to check that X^2 in Example 3.1 is a Lindelöf space. Therefore, we have the following question:

Question 3.4 Is the product of two charming spaces Lindelöf ? In particular, is the product of a charming space with a Lindelöf Σ-space a Lindelöf space?

Finally, we discuss a charming space with a G_δ-diagonal.

Theorem 3.1[2] *A Lindelöf Σ-space with a G_δ-diagonal has a countable network.*

Proposition 3.6 *Let S be the Sorgenfrey line. Then any uncountable subspace of S is not a Lindelöf Σ-subspace.*

Proof. Let Y be an uncountable subspace of S. Assume that Y is a Lindelöf Σ-subspace. Then $L = Y \bigcup (-Y)$ is also a Lindelöf Σ-subspace of S, hence L^2 has a countable network by Theorem 3.1 since S has a G_δ-diagonal. However, L^2 contains a closed, uncountable and discrete subspace $\{(x, -x) : x \in L\}$, which is a contradiction since L^2 has a countable network.

Example 3.2 There exists a hereditarily Lindelöf space X which is not a charming space.

Proof. Let X be the Sorgenfrey line. Then X is a hereditarily Lindelöf space. Assume that X is a charming space. Then there exists a subspace Y such that Y is a $(\mathscr{P}_4, \mathscr{P}_4)$-shell in X. By Proposition 3.6, Y is countable. Let $Y = \{b_n : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, take an open neighborhood $[b_n, b_n + 2^{-n})$ in X. Then

$$U = \bigcup_{n\in\mathbf{N}} [b_n, b_n + 2^{-n})$$

is an open neighborhood of Y in X. Since

$$m(Y) \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

we have that $X \setminus U$ is an uncountable set. Then $X \setminus U$ is not a Lindelöf Σ-space by Proposition 3.6, which is a contradiction.

It is natural to ask the following question.

Question 3.5 Does each charming space X with a G_δ-diagonal have a countable network?

By Theorem 3.1, the following proposition is obvious.

Proposition 3.7 *Each charming space X with a G_δ-diagonal is a $(2,2)$-structured space.*

The following two results are well-known in the class of Lindelöf Σ-spaces.

Theorem 3.2[6] *Each hereditarily Lindelöf Σ-space has a countable network.*

Theorem 3.3[5] *Each Lindelöf Σ-space with a point-countable base is second-countable.*

However, the following questions are remain open.

Question 3.6 Is each hereditarily charming space a Lindelöf Σ-space?

Question 3.7 Does each hereditarily charming space have a countable network?

Question 3.8 Is each charming space with a point-countable base metrizable?

4 CL-charming and CO-charming Spaces

In this section, we introduce two classes of charming spaces, and then discuss some properties of them.

Definition 4.1 *A space X is called CL-charming space (resp., CO-charming space) if there is a closed subspace (resp., compact subspace) Y of X such that Y is a $(\mathscr{P}_4, \mathscr{P}_4)$-shell in X.*

Obviously, each CO-charming space is CL-charming, and each CL-charming is a charming space. The space X in Example 3.1 is CO-charming. However, there exists a CL-charming space is not CO-charming space.

Example 4.1 There exists a CL-charming space which is not CO-charming.

Proof. For each $n \in \mathbf{N}$, let X_n be the copy of X in Example 3.1. Then it is easy to see that the topological sum $Y = \bigoplus\limits_{n\in\mathbf{N}} X_n$ is a CL-charming space. However, it is not a CO-charming space.

The following three propositions are easy to check.

Proposition 4.1 *Any image of a CO-charming space under a continuous mapping is a CO-charming space.*

Proposition 4.2 *Any closed image of a CL-charming space under a continuous mapping is a CL-charming space.*

Proposition 4.3 *Any preimage of a CL-charming space (resp., CO-charming space) under a perfect mapping is a CL-charming space (resp., CO-charming space).*

However, we do not know the answer of the following question.

Question 4.1 Is any image of a CL-charming space under a continuous mapping a CL-charming space?

Theorem 4.1 *A hereditarily CL-charming space X is a hereditarily Lindelöf Σ-space. Therefore, X has a countable network.*

Proof. Let X be a hereditarily CL-charming space. Take a closed subspace Y of X such that Y is a Lindelöf Σ-space. Obviously, X is hereditarily Lindelöf, hence Y has a countable pseudocharacter in X. Put $Y = \bigcap\limits_{k\in\omega} U_k$, where each U_k is an open neighborhood in X. Since every $X \setminus U_k$ is a Lindelöf Σ-space, we have that $\bigcup\limits_{n\in\omega} (X \setminus U_n)$ is a Lindelöf Σ-space. Therefore,

$$\bigcup_{n\in\omega} (X \setminus U_n) = X \setminus \bigcap_{n\in\omega} U_n = X \setminus Y$$

is a Lindelöf Σ-space. Then $X = (X \setminus Y) \bigcup Y$ is a Lindelöf Σ-space. Therefore, X is a hereditarily Lindelöf Σ-space, which implies that X has a countable network (see Corollary 4.13 in [6]).

Theorem 4.2 *A CO-charming space X with a point-countable base is metrizable.*

Proof. Let $\mathcal{B}$ be a point-countable base of X. Since X is CO-charming, there exists a compact subspace K of X such that K is a $(\mathscr{P}_4, \mathscr{P}_4)$-shell in X. Moreover, since a compact space with a point-countable base is metrizable, K is a separable and metrizable space. Put $\mathcal{B}' = \{B \in \mathcal{B} : B \bigcap K \neq \emptyset\}$. Then $\mathcal{B}'$ is countable since K is separable. Let

$$\mathcal{U} = \{\bigcup \mathcal{F} : \mathcal{F} \subset \mathcal{B}',\ K \subset \bigcup \mathcal{F},\ |\mathcal{F}| < \omega\}.$$

Then it is easy to check that $\mathcal{U} = \{U_n : n \in \mathbf{N}\}$ is a countable base of K in X. For each $n \in \mathbf{N}$, let $X_n = X \setminus U_n$. Since each X_n is a Lindelöf Σ-subspace with a point-countable base, each X_n is separable and metrizable by Theorem 3.3. Therefore,

$$X = K \bigcap (X \setminus K) = K \bigcap (X \setminus \bigcap_{n\in\mathbf{N}} U_n) = K \bigcap \bigcup_{n\in\mathbf{N}} (X \setminus U_n),$$

which implies that X is separable. Since a separable space with a point-countable base has a countable base, X is metrizable.

The following result gives a partial answer to Question 3.5.

Theorem 4.3 *A CO-charming space with a G_δ-diagonal has a countable network.*

Proof. Let X be a CO-charming space. Then there exists a compact subspace K of X such that K is a $(\mathscr{P}_4, \mathscr{P}_4)$-shell in X. Since X has G_δ-diagonal, K is a G_δ-set in X by Proposition 2.3 in [8]. Put $K = \bigcap\limits_{n\in\omega} U_n$, where each U_n is open in X. For each $n \in \mathbf{N}$, let $X_n = X \setminus U_n$. Then each X_n is a Lindelöf Σ-subspace in X. Hence

$$X \setminus K = X \setminus (\bigcap_{n\in\omega} U_n) = \bigcup_{n\in\omega} X \setminus U_n = \bigcup_{n\in\omega} X_n$$

is a Lindelöf Σ-subspace. Therefore, $X = K \bigcup (X \setminus K)$ is a Lindelöf Σ-space. By Theorem 4 in [2], X has a countable network.

5 The Characterizations of Weak (i, j)-structured Spaces

Definition 5.1 *For any $i, j \in \{0, 1, 2, 3, 4, 5\}$, a space X is called a weak (i, j)-structured space if there exists a space Y which maps continuously onto X and perfectly onto a (i, j)-structured space.*

Obviously, each weak $(1, 1)$-structured space is a charming space, and hence it is a Lindelöf space. However, the following question is still open.

Question 5.1 Is each charming space weak $(1, 1)$-structured?

Proposition 5.1 *Each closed subspace F of a weak (i, j)-structured space X is also a weakly (i, j)-structured space.*

Proof. Take a space Y for which there exists a continuous onto map $\varphi : Y \to X$ and a perfect map $h : Y \to M$ for some (i, j)-structured space M. Let F be a closed subspace of Y and let $Z = \varphi^{-1}(F)$. Then F is a continuous image of Z and it is easy to see that $h \mid Z : Z \to M$ is a perfect map. Hence F is a weak (i, j)-structured space.

The following theorem gives a characterization of weak (i, j)-structured spaces. First, we recall some concepts.

Any map φ from a space X to the family $\exp\{Y\}$ of subsets of Y is called multivalued; for convenient, we always write φ: $X \to Y$ instead of φ: $X \to \exp\{Y\}$. A multivalued map φ: $X \to Y$ is called compact-valued (resp., finite-valued) if the set $\varphi(x)$ is compact (resp., finite) for each $x \in X$.

Let φ: $X \to Y$ be a multivalued map. For any $A \subset X$, we denote by $\varphi(A)$ the set $\bigcup\{\varphi(x) : x \in A\}$, that is, $\varphi(A) = \bigcup\{\varphi(x) : x \in A\}$; we say that the map φ is onto if $\varphi(X) = Y$. The map φ is called upper semicontinuous if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$ is open in X for any open subset U in Y.

Theorem 5.1 *For arbitrary $i, j \in \{0, 4, 5\}$, the following conditions are equivalent for any space X:*

(1) *X is a weak (i, j)-structured space*;

(2) *there exist spaces K and M such that K is compact, M is an (i, j)-structured space and X is a continuous images of a closed subspace of $K \times M$*;

(3) *X belongs to any class $\mathscr{P}$ which satisfies the following conditions*:

(a) *$\mathscr{P}$ contains compact spaces and (i, j)-structured spaces*;

(b) *$\mathscr{P}$ is invariant under closed subspaces and continuous images*;

(c) *$\mathscr{P}_5 \times \mathscr{P}$ is contained in $\mathscr{P}$*;

(4) *there is an upper semicontinuous compact-valued onto a map $\varphi : M \to X$ for some (i, j)-structured space M.*

Proof. (1)$\Rightarrow$(2). There exists a space Y which maps continuously onto X and perfectly onto an (i, j)-structured space M. Then we fix the respective perfect map $h : Y \to M$. Let

$i : Y \to \beta Y$ be the identity embedding. Then the diagonal map $g = h\Delta i : Y \to M \times \beta Y$ is perfect by Theorem 3.7.9 in [4], and so the set $g(Y)$ is closed in $M \times \beta Y$. Since g is injective, $g(Y)$ is homeomorphic to Y, and hence X is a continuous image of $g(Y)$.

(2)⇒(3). Suppose that X is a continuous image of a closed subset F of the product $M \times K$ for some (i, j)-structured space M and compact space K. Take any class $\mathscr{P}$ as in (2). Then $K \in \mathscr{P}$ and $M \in \mathscr{P}$. Therefore $M \times K \in \mathscr{P}$ and hence $F \in \mathscr{P}$ as well since $\mathscr{P}$ is invariant under closed subspaces. Since $\mathscr{P}$ is invariant under continuous images, we have $X \in \mathscr{P}$.

(3)⇒(1). First, we show if X satisfies (1) then any closed subspace F of X also satisfies (1). Indeed, take a space Y for which there exists a continuous onto map $\varphi : Y \to X$ and a perfect map $h : Y \to M$ for some (i, j)-structured space M. Let F is an any closed subspace of X and let $Z = \varphi^{-1}(F)$. Then F is a continuous image of Z and it is easy to see that $h \mid Z : Z \to M$ is a perfect map. Hence F satisfies (1). It is evident that the class of spaces satisfies (1) is invariant under continuous images. Moreover, the classes of (i, j)-structured spaces and compact spaces satisfy (1).

(1)⇒(4). Suppose that X satisfies (1), i.e., there exists a space Y which maps continuously onto X and perfectly onto an (i, j)-structured space N. Fix the respective map $f : Y \to X$ and a perfect map $g : Y \to N$. Let $M = g(Y)$. For every $x \in X$, since the set $\varphi(x) = f(g^{-1}(x)) \subset X$ is compact, $\varphi : M \to X$ is a compact-valued map. The map f is surjective, and hence $\varphi(M) = X$. It is easy to see that φ is upper semicontinuous.

(4)⇒(2). Fix an (i, j)-structured space M and a compact-valued upper semicontinuous onto map $\varphi : M \to X$. Put $F = \bigcup\{\varphi(x) \times \{x\} : x \in M\}$. Then F is contained in $\beta X \times M$. Let $\pi : \beta X \times M \to \beta X$ be the projection. Then $\pi(F) = X$ so X is a continuous image of F. Fix any point $(x, t) \in (\beta X \times M) \setminus F$. Then $x \notin \varphi(t)$, and hence we can find disjoint sets $U, V \in \tau(\beta X)$ such that $x \in U$ and $\varphi(t) \subset V$. Since φ is upper semicontinuous, there exists a set $W \in \tau(t, M)$ such that $\varphi(W) \subset V$. It easily check that $(x, t) \in U \times W \subset (\beta X \times M) \setminus F$, therefore the set F is closed in $\beta X \times M$.

6 Rectifiable Spaces with a Weak (i, j)-structure

Recall that a topological group G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A topological space G is said to be a rectifiable space provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a rectification on G. It is well known that rectifiable spaces are a good generalizations of topological groups. In fact, for a topological group with the neutral element e, then it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, the 7-dimensional sphere S_7 is rectifiable but not a topological group (see [8]).

Theorem 6.1[9]–[11] *A topological space G is rectifiable if and only if there exist a $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x, y \in G$ the next identities hold*:

$$p(x, q(x, y)) = q(x, p(x, y)) = y \quad \textit{and} \quad q(x, x) = e.$$

Given a rectification φ of the space G, we may obtain the mappings p and q in Theorem 6.1 as follows. Let $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$. Then the mappings p and q satisfy the identities in Theorem 6.1, and both are open mappings.

Let G be a rectifiable space, and p be the multiplication on G. Further, we sometimes write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$, since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with operations p and q, a 0-ary operation e, and identities as above. It is easy to see that this algebraic system need not to satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

Lemma 6.1 *If A is a subset of rectifiable space G, then $H = \bigcup_{n\in\mathbf{N}} (A_n \bigcup B_n)$ is also a rectifiable subspace of G, where*

$$\begin{aligned}
&A_1 = A, && B_1 = q(A, e) \bigcup q(A, A),\\
&A_2 = p(A_1 \bigcup B_1, A_1 \bigcup B_1), && B_2 = q(A_1 \bigcup B_1, A_1 \bigcup B_1),\\
&A_{n+1} = p(A_n \bigcup B_n, A_n \bigcup B_n), && B_{n+1} = q(A_n \bigcup B_n, A_n \bigcup B_n), \qquad n = 1, 2, \cdots
\end{aligned}$$

Obviously, if A is a Lindelöf Σ-subspace, then H is a Lindelöf Σ-space.

Proof. Since $q(A, A) \subset B_1$, we have $e \in B_1$. Therefore, it is easy to see that

$$A_n \bigcup B_n \subset A_{n+1} \bigcup B_{n+1}, \qquad n \in \mathbf{N}.$$

Put

$$H = \bigcup_{n\in\mathbf{N}} (A_n \bigcup B_n).$$

Next we shall prove that H is a rectifiable subspace of G. Indeed, take arbitrary points $x, y \in B$. Then there exists an $n \in \mathbf{N}$ such that $x, y \in A_n \bigcup B_n$, and hence

$$p(x, y) \in A_{n+1} \bigcup B_{n+1}, \qquad q(x, y) \in A_{n+1} \bigcup B_{n+1}.$$

Therefore, H is a rectifiable subspace of G.

Since the product of a countable family of Lindelöf Σ-space is Lindelöf Σ, it is easy to see that H is Lindelöf Σ.

Theorem 6.2 *Every charming rectifiable space G has a dense rectifiable subspace that is a Lindelöf Σ-space.*

Proof. Since G is charming, there exists a subspace B such that G is a $(\mathscr{P}_4, \mathscr{P}_4)$-shell of G.

Case 1. B is dense in G.

Take the smallest rectifiable subspace H of G such that $L \subset G$. Then H is also a Lindelöf Σ-space by Lemma 6.1. Clearly, H is dense in G.

Case 2. B is not dense in G.

Then there exists a non-empty open subset U of G such that $\overline{U} \bigcap B = \emptyset$. Put $V = G \setminus \overline{U}$ is an open neighborhood of B, and hence $\overline{U}$ is a Lindelöf Σ-space. Then $\mathscr{A} = \{x \cdot \overline{U} : x \in G\}$ is an open cover of G, and each element of $\mathscr{A}$ is homeomorphic to $\overline{U}$. Since G is Lindelöf, there exists a countable subcover of $\mathscr{A}$, and hence G is a Lindelöf Σ-space.

A topological space X has the Suslin property if every pairwise disjoint family of non-empty open subsets of X is countable.

Lemma 6.2[12] *The Suslin number of an arbitrary Lindelöf Σ-rectifiable space G is countable.*

Theorem 6.3 *The Suslin number of an arbitrary charming rectifiable space G is countable.*

Proof. By Theorem 6.2, G has a dense rectifiable subspace H which is a Lindelöf Σ-subspace. Then the Suslin number of H is countable by Lemma 6.2. Since H is dense in G, it follows that the Suslin number of G is countable.

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